NEW RESULTS ON THE OPERATOR CARLESON MEASURE CRITERION

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ABSTRACT. We consider control systems of the form $\dot{x} = Ax + Bu$ where A is the generator of a diagonal semigroup $\mathbb{T}$ on l^2 and B is an unbounded operator from a Hilbert space U to l^2. In a previous paper by Hansen and Weiss, a condition called the operator Carleson measure criterion was shown to be necessary for the admissibility of the control operator B. Furthermore this condition was shown to be sufficient if $\mathbb{T}$ is either analytic or invertible. In this paper we continue the analysis of admissibility as related to the operator Carleson measure criterion. We show that the operator Carleson measure criterion is satisfied if and only if the input-to-state transfer function has a certain decay rate. We also extend the previous sufficiency results of Hansen and Weiss to a more general class of diagonal semigroups. To achieve our aims, we derive some general results (not confined to diagonal semigroups) concerning Lyapunov equations and feedback type perturbations, which are of independent interest.

1. Introduction and statement of the main results.

In the paper Hansen and Weiss [8], a necessary condition for admissibility of control operators for diagonal semigroups on l^2 was proved. This condition, called the operator Carleson measure criterion, was shown to be a sufficient condition as well, if the semigroup is either analytic or invertible. In this paper we continue this analysis of the operator Carleson measure criterion and its relationship to admissible control operators. We prove that a system satisfies this criterion if and only if its input-to-state transfer function possesses a certain decay rate. We also show that the results of [8] concerning analytic or invertible semigroups can be extended to a larger class of diagonal semigroups.

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Some of our intermediate results relate to arbitrary (nondiagonal) strongly continuous semigroups and are interesting in their own right. We have in mind the connection between admissibility, stability and the Lyapunov equation, as well as the invariance of admissibility under (possibly unbounded) feedback.

All vector spaces considered here are complex. We represent elements of l^2 as infinite column matrices (i.e., matrices consisting of a single column) and elements of the dual l^{2*} as infinite row matrices.

We are concerned with control systems described by

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1.1}$$

where $x(t) \in l^2$ is the state and $u \in L^2([0,\infty), l^2)$ is the input function. The operators A and B are represented by infinite matrices. A is diagonal and its diagonal elements λ_k (its eigenvalues) are in the open left half of the complex plane:

$$-\lambda_k \in \mathbb{C}_0 \qquad\qquad \forall\, k \in \mathbb{N}, \tag{1.2}$$

where

$$\mathbb{C}_0 = \{\, s \in \mathbb{C} \mid \ \mathrm{Re}\ s > 0 \,\}.$$

Therefore A generates a strongly continuous diagonal semigroup $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ on l^2:

$$(\mathbb{T}_t\, z)_k = e^{\lambda_k t} z_k, \qquad\qquad \forall\, k \in \mathbb{N}, \tag{1.3}$$

where z_k denotes the k-th component of $z \in l^2$. The infinite matrix B may be unbounded in the sense that its range, when applied to l^2, is not necessarily contained in l^2. B represents an *admissible control operator* for $\mathbb{T}$ if for any input function $u \in L^2([0,\infty), l^2)$, (1.1) together with the initial condition $x(0) = 0$ has an l^2-valued strong solution $x(\cdot)$. If this is the case then $x(\cdot)$ is a continuous function from $[0, \infty)$ to l^2 (for more details on this see Section 2). If in addition, the function $x(\cdot)$ is bounded (for every u) then B is called *infinite-time admissible* for $\mathbb{T}$.

The assumption that $\mathbb{T}$ is diagonal is restrictive in that not all semigroups are isomorphic to diagonal ones. However, the stability assumption in (1.2) is imposed mainly to avoid trivialities. All the results in this paper concerning admissibility could be reworded so that (1.2) would not be needed. Our choice of l^2 for the input space also leads to no loss of generality (see Remark 2.7).

To recall the results of [8], we need a notation for certain rectangles in $\mathbb{C}$:

$$R(h,\omega) = \left\{ z \in \mathbb{C} \mid 0 < \ \mathrm{Re}\ z \leq h, \quad \omega - h \leq \ \mathrm{Im}\ z < \omega + h \right\}, \tag{1.4}$$

for any $h + i\omega \in \mathbb{C}_0$ $(i = \sqrt{-1})$. We denote by b_k the k-th row of B. Then $b_k^* b_k$ is an infinite matrix of rank one.

Definition 1.1. The infinite matrix B, with rows b_k, satisfies the *operator Carleson measure criterion* for the semigroup $\mathbb{T}$ defined by (1.3) if $b_k \in l^{2*}$ for any $k \in \mathbb{N}$ and there is some $M \geq 0$ such that for any $h + i\omega \in \mathbb{C}_0$,

$$\Bigg\| \sum_{-\lambda_k \in R(h,\omega)} b_k^* b_k \Bigg\|_{\mathcal{L}(l^2)} \leq Mh\,. \tag{1.5}$$

We denote by $\mathrm{OCM}(\mathbb{T})$ the space of infinite matrices which satisfy the operator Carleson measure criterion for $\mathbb{T}$. The following two results concerning the operator Carleson measure criterion were proved in [8].

Theorem 1.2. *Assume $\mathbb{T}$ defined by* (1.3) *is exponentially stable and the infinite matrix B represents an admissible control operator for $\mathbb{T}$. Then $B \in \mathrm{OCM}(\mathbb{T})$.*

Theorem 1.3. *Assume $\mathbb{T}$ defined by* (1.3) *is exponentially stable and either analytic or invertible. If $B \in \mathrm{OCM}(\mathbb{T})$ then B represents an admissible control operator for $\mathbb{T}$.*

Remark 1.4. By modifying the definition of the rectangles $R(h,\omega)$ in (1.4) (by left translation), (1.5) can be used to check admissibility in cases where the semigroup is not assumed to be exponentially stable. Furthermore, to use the operator Carleson measure criterion, it is not necessary to verify (1.5) for *all* $h + i\omega \in \mathbb{C}_0$; see Proposition 6.1. It is also worth noting that the rectangles $R(h,\omega)$ defined in (1.4) could have been replaced by the analogous set of translations and dilations of *any* rectangle, or of a half-disk, or certain other sets, without changing the meaning of Definition 1.1. (Our choice of 2×1 rectangles follows Ho and Russell [9].)

Our main results are in Sections 5 and 6. The following theorem lists some of their consequences, thus indicating the general direction of this paper.

Theorem 1.5. *Assume that the semigroup $\mathbb{T}$ is given by* (1.2), (1.3) *and B is an infinite matrix with rows b_k in l^{2*}.*

(i) *B is an infinite-time admissible control operator for $\mathbb{T}$ if and only if the infinite matrix $P = (p_{jk})$, whose entries are given by*

$$p_{jk} = -\frac{\langle b_j, b_k\rangle}{\lambda_j + \overline{\lambda}_k}$$

represents a bounded operator on l^2.

(ii) *$B \in \mathrm{OCM}(\mathbb{T})$ if and only if there exists a $K \geq 0$ for which*

$$\sup_{\|\mathrm{v}\|_{l^2} \leq 1} \sum_{k=1}^{\infty} \left| \frac{b_k \mathrm{v}}{s - \lambda_k} \right|^2 \leq \frac{K}{\operatorname{Re} s}, \qquad \forall\, s \in \mathbb{C}_0. \tag{1.6}$$

(iii) *Suppose that there are numbers $0 < a \leq b$ and $\alpha \geq 0$ such that*

$$a|\operatorname{Im} \lambda_k|^{\alpha} \leq -\operatorname{Re} \lambda_k \leq b|\operatorname{Im} \lambda_k|^{\alpha}. \tag{1.7}$$

Then $B \in \mathrm{OCM}(\mathbb{T})$ if and only if B is an infinite-time admissible control operator for $\mathbb{T}$.

Some comments about this theorem are in order.

Concerning (i), the infinite matrix P is the *controllability Gramian* for (1.1). If P is bounded then it is the unique self-adjoint solution of a Lyapunov equation, see Section 5. The equivalence between infinite-time admissibility and boundedness of the controllability Gramian holds also for non-diagonal semigroups, see Theorem 3.1.

Concerning (ii), when B satisfies the operator Carleson measure criterion for $\mathbb{T}$, the left-hand side of (1.6) is the norm squared of the input–to–state transfer function, i.e., $\|(sI-A)^{-1}B\|^2_{\mathcal{L}(l^2)}$ (see Section 5). The condition $\|(sI-A)^{-1}B\|^2 \le K/\mathrm{Re}\, s$ is necessary for the infinite-time admissibility of B also for nondiagonal semigroups, see Section 2. It has been conjectured in [21] that it is sufficient as well. (In [21] it is assumed that $\mathbb{T}$ is exponentially stable, but this difference is not significant.)

A result similar to (iii) was announced in [8, Section 5]. The geometric restriction on the position of the eigenvalues in (1.7) can be substantially relaxed such that it is almost always satisfied in physically motivated systems of the form (1.1)–(1.3); see Proposition 6.9. It is unknown whether or not the equivalence in (iii) remains true without any geometric restriction on the eigenvalues (see [8, Conjecture 4.4]).

This paper is organized as follows. In Section 2 we provide background on admissible and infinite-time admissible control operators. In Section 3 we consider general systems of the type (1.1), without the restrictive assumption that A is diagonal. In this setting, we prove that infinite-time admissibility is equivalent to the existence of the controllability Gramian, which in turn is equivalent to solvability of a Lyapunov equation (see Theorem 3.1 for the precise statement). In Section 4 we prove the feedback invariance of admissiblility (this is needed for a proof in Section 6). In Section 5 we prove a stronger version of the results in Section 3 for the case where A is diagonal and also prove (i) and (ii) of Theorem 1.5. In Section 6 we extend the results of [8] to other classes of diagonal semigroups. In particular, we prove part (iii) of Theorem 1.5.

2. Some background.

In this section we give some general facts about admissible and infinite-time admissible control operators, following Hansen and Weiss [8], Ho and Russell [9], Salamon [14], [15] and Weiss [17], [21] (our notation follows [8] and [17]).

We need notation for some spaces which will be used. Suppose that A is the generator of a strongly continuous semigroup $\mathbb{T} = (\mathbb{T}_t)_{t\ge 0}$ on the Hilbert space X. For any $n \in \mathbb{N}$ define the space X_{-n} as the completion of X with respect to the norm

$$\|x\|_{-n} = \|(\beta I - A)^{-n} x\|,$$

where $\beta \in \rho(A)$ (X_{-n} does not depend upon β). We put $X_0 = X$. Then $(\beta I - A)^{-1}$ extends to an isomorphism from X_{-n} to X_{-n+1}. $\mathbb{T}$ extends to a strongly continuous semigroup on X_{-n}, whose generator is an extension of A, with domain X_{-n+1}. The extended semigroup is isomorphic to the initial one. We denote the extensions of $\mathbb{T}$ and A by the same symbols. Let Z_n be the Hilbert space obtained by endowing $D((A^*)^n)$ with the norm

$$\|x\|_n = \|(\beta I - A^*)^n x\|.$$

We identify X with X^*. It follows that $Z_n^* = X_{-n}$ for any $n \in \mathbb{N}$. For $z \in Z_n$ and $x \in X_{-n}$, we denote by $\langle z, x\rangle$ the duality pairing which reduces to the usual scalar product on X if $x \in X$.

Definition 2.1. With the above notation, let U be a Hilbert space and $B \in \mathcal{L}(U, X_{-1})$. Then B is said to be an *admissible control operator* for $\mathbb{T}$, if for some $\tau > 0$ and any $u \in L^2([0,\infty), U)$ we have $\Phi_\tau u \in X$, where $\Phi_\tau u$ is defined by

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma} B u(\sigma)\, d\sigma\,.$$

If B is admissible then for any $\tau \geq 0$, Φ_τ defined above is a bounded linear operator from $L^2([0,\infty), U)$ to X (this follows from the closed graph theorem). In other words, for each $\tau \geq 0$ there is a $k_\tau \geq 0$ such that

$$\|\Phi_\tau u\|_X \leq k_\tau \|u\|_{L^2} \qquad \forall\, u \in L^2([0,\infty), U). \tag{2.1}$$

The concept of admissibility is important because it is equivalent to the solvability, in a reasonable sense, of the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{2.2}$$

More precisely, if B is admissible, then for any $x_0 \in X$ and any $u \in L^2_{loc}([0,\infty), U)$, the X-valued function x defined on $[0,\infty)$ by

$$x(t) = \mathbb{T}_t x_0 + \Phi_t u \tag{2.3}$$

is continuous (in X), and it is a strong solution of (2.2) (in X_{-1}). Any *abstract linear control system* may be represented in the form (2.2), with admissible $B \in \mathcal{L}(U, X_{-1})$, see [16], [17] for the definition and for details.

The space $\mathcal{B}(U, X, \mathbb{T})$ of all admissible control operators for $\mathbb{T}$ with domain U is a subspace of $\mathcal{L}(U, X_{-1})$. This space becomes a Banach space with the norm

$$|||B|||_\tau = \|\Phi_\tau\|_{\mathcal{L}(L^2,X)}\,,$$

where the choice of $\tau > 0$ is unimportant for the topology of $\mathcal{B}(U, X, \mathbb{T})$.

Definition 2.2. With the above notation, an operator $B \in \mathcal{L}(U, X_{-1})$ is *infinite-time admissible* for $\mathbb{T}$ if $B \in \mathcal{B}(U, X, \mathbb{T})$ and for any $u \in L^2([0,\infty), U)$, the function $\tau \to \Phi_\tau u$ (from $[0,\infty)$ to X) is bounded.

If B is infinite-time admissible then the constant k_τ appearing in (2.1) can be chosen to be independent of τ (this follows from the uniform boundedness theorem):

$$\|\Phi_\tau u\|_X \leq k\|u\|_{L^2} \qquad \forall\, u \in L^2([0,\infty), U). \tag{2.4}$$

We denote by $\tilde{\mathcal{B}}(U, X, \mathbb{T})$ the space of all infinite-time admissible control operators for $\mathbb{T}$ with domain U. $\tilde{\mathcal{B}}(U, X, \mathbb{T})$ becomes a Banach space with the norm

$$|||B|||_\infty = \lim_{\tau\to\infty} |||B|||_\tau\,.$$

(The completeness of this space follows from the completeness of $\mathcal{B}(U,X,\mathbb{T})$.) If the semigroup $\mathbb{T}$ is exponentially stable then $\tilde{\mathcal{B}}(U,X,\mathbb{T})=\mathcal{B}(U,X,\mathbb{T})$. Otherwise (even if $\mathbb{T}$ is strongly stable) the two notions of admissibility are not equivalent; see Remark 2.8.

We denote by $\tilde{\Phi}_\tau$ the operator obtained from Φ_τ by reversing the time on $[0,\tau]$:

$$\tilde{\Phi}_\tau u = \int_0^\tau \mathbb{T}_\sigma\, Bu(\sigma)\, d\sigma\,.$$

Clearly $\|\tilde{\Phi}_\tau\| = \|\Phi_\tau\|$ holds. A simple but important fact is that $B\in\tilde{\mathcal{B}}(U,X,\mathbb{T})$ if and only if there exists $\tilde{\Phi}\in\mathcal{L}(L^2([0,\infty),U),X)$ such that for every $u\in L^2([0,\infty),U)$

$$\tilde{\Phi}u = \lim_{\tau\to\infty}\tilde{\Phi}_\tau u \qquad (\text{in } X)\,. \tag{2.5}$$

In this formula, by writing "in X" we mean that the limit converges in X. In order to prove (2.5), we use (2.4) to show that for $0\le\tau\le t$

$$\|\tilde{\Phi}_t u - \tilde{\Phi}_\tau u\|_X \;\le\; k\|u\|_{L^2([\tau,t],U)}\,.$$

Clearly $\|\tilde{\Phi}\|\le k$, k being the constant appearing in (2.4).

Let $\delta\ge 0$ be such that for any $\beta>\delta$, $e^{-\beta t}\|\mathbb{T}_t\|\to 0$ as $t\to\infty$. Let $\mathbb{C}_\delta$ denote the set of complex numbers s with $\text{Re}\ s>\delta$. By taking in (2.5) $u(t)=\mathrm{v}e^{-st}$, where $\mathrm{v}\in U$ and $s\in\mathbb{C}_\delta$, and estimating $\|\tilde{\Phi}u\|$, we obtain that for $K=k^2/2$

$$\|(sI-A)^{-1}B\|^2_{\mathcal{L}(U,X)} \;\le\; \frac{K}{\text{Re}\ s}\,, \qquad \forall\, s\in\mathbb{C}_\delta\,.$$

Thus, the above estimate is a necessary condition for the infinite-time admissibility of B. It has been conjectured in Weiss [21] that it is sufficient as well, and various partial results in this direction have been obtained in [8] and [21] (these papers assume that the semigroup is exponentially stable, but this is not a significant restriction). Related material and extensions are contained in the recent papers of Grabowski [5] and Grabowski and Callier [6].

We give now the dual formulation of the concepts introduced above. For any $B\in\mathcal{L}(U,X_{-1})$ and any $\tau>0$, the dual of $\tilde{\Phi}_\tau$ is an operator in $\mathcal{L}(Z_1,L^2([0,\infty),U))$ (we make the identification $U=U^*$) which is given by

$$(\tilde{\Phi}^*_\tau x)(t) \;=\; \begin{cases} B^*\mathbb{T}^*_t x & t\le\tau, \\ 0 & t>\tau, \end{cases} \qquad \forall\, x\in Z_1\,.$$

It follows that $B\in\mathcal{B}(U,X,\mathbb{T})$ if and only if for some (hence for any) $\tau>0$, $\tilde{\Phi}^*_\tau$ extends continuously to X. In other words, there is a $k_\tau\ge 0$ such that

$$\int_0^\infty \|(\tilde{\Phi}^*_\tau x)(t)\|^2_U\, dt \;\le\; k^2_\tau\|x\|^2_X\,, \qquad \forall\, x\in Z_1\,.$$

We have $B\in\tilde{\mathcal{B}}(U,X,\mathbb{T})$ if and only if B is admissible and the constant k_τ appearing above can be chosen to be independent of τ. Assume that $B\in\mathcal{L}(U,X_{-1})$ and define for every $x\in Z_1$ the function Ψx on $[0,\infty)$ by

$$(\Psi x)(t) \;=\; B^*\mathbb{T}^*_t x\,, \qquad \forall\, x\in Z_1\,.$$

It is now clear that $B \in \tilde{\mathcal{B}}(U, X, \mathbb{T})$ if and only if there is a $k \geq 0$ (in fact the same as in (2.4)) such that

$$\int_0^\infty \|(\Psi x)(t)\|_U^2 \, dt \, \leq \, k^2 \|x\|_X^2 \,, \qquad \forall \, x \in Z_1. \tag{2.6}$$

Equivalently, $B \in \tilde{\mathcal{B}}(U, X, \mathbb{T})$ if and only if Ψ has an extension to X which is bounded as an operator from X to $L^2([0, \infty), U)$. This extension, still denoted Ψ, is the adjoint of the operator $\tilde{\Phi}$ defined in (2.5):

$$\Psi \, = \, \tilde{\Phi}^* \, . \tag{2.7}$$

A formula for Ψ which is valid on X will be given in Remark 3.6.

When $U = \mathbb{C}$, the spaces $\mathcal{B}(\mathbb{C}, X, \mathbb{T})$ and $\tilde{\mathcal{B}}(\mathbb{C}, X, \mathbb{T})$ can be identified with subspaces of X_{-1}, which we denote by $\mathfrak{b}(X, \mathbb{T})$ and by $\tilde{\mathfrak{b}}(X, \mathbb{T})$. Clearly $\tilde{\mathfrak{b}}(X, \mathbb{T}) \subset \mathfrak{b}(X, \mathbb{T})$. In the remainder of this section we consider diagonal $\mathbb{T}$, given by (1.2) and (1.3). The following is a slight variation of a theorem of Ho and Russell [9] and Weiss [18].

Theorem 2.3. *Assume $\mathbb{T}$ is given by (1.2), (1.3) and $b = (b_k)_{k \in \mathbb{N}}$ is a sequence of complex numbers. Then $b \in \tilde{\mathfrak{b}}(l^2, \mathbb{T})$ if and only if there exists $M > 0$ such that for all $h + i\omega \in \mathbb{C}_0$*

$$\sum_{-\lambda_k \in R(h,\omega)} |b_k|^2 \, \leq \, Mh \, . \tag{2.8}$$

We have $b \in \mathfrak{b}(l^2, \mathbb{T})$ if and only if for some (hence, for every) $\delta > 0$ there exists an $M > 0$ (M may depend on δ) such that (2.8) holds for all $h + i\omega \in \mathbb{C}_\delta$.

Proof. The first part of Theorem 2.3 was proved for the case where $\mathbb{T}$ is exponentially stable in [9] (the 'if' part) and [18] (the 'only if' part). We leave it to the reader to verify that the same proofs remain valid under the weaker hypothesis (1.2).

To prove the second part, replace $\mathbb{T}_t$ by $\tilde{\mathbb{T}}_t = e^{-\delta t}\mathbb{T}_t$ and use the fact that

$$\mathfrak{b}(l^2, \mathbb{T}) = \mathfrak{b}(l^2, \tilde{\mathbb{T}}) = \tilde{\mathfrak{b}}(l^2, \tilde{\mathbb{T}}) \, .$$

Thus, by the first part of the theorem, b is admissible for $\mathbb{T}$ iff there exists an $M > 0$ such that for all $h + i\omega \in \mathbb{C}_0$

$$\sum_{-\lambda_k \in R(h,\omega) - \delta} |b_k|^2 \, \leq \, Mh \, . \tag{2.9}$$

Only $h > \delta$ is relevant in (2.9), since otherwise $R(h, \omega) - \delta$ does not intersect $\mathbb{C}_0$. It is clear that (2.8) on $\mathbb{C}_\delta$ (i.e., for $h > \delta$) implies (2.9). Conversely, (2.9) implies that (2.8) holds on $\mathbb{C}_\delta$, with Mh replaced by $M(h + \delta)$, which is less than $2Mh$. $\square$

The elements of $\tilde{\mathfrak{b}}(l^2, \mathbb{T})$ are called *infinite-time admissible input elements* and the inequality (2.8) is called the *Carleson measure criterion* for the infinite-time admissibility of input elements. It is easy to see that the operator Carleson measure criterion (1.5) reduces to (2.8) when the infinite matrix B has only one column.

Conjecture 4.4 from [8] can be slightly generalized, to include semigroups which are not necessarily exponentially stable. With the notation introduced earlier, this generalized conjecture states that for diagonal semigroups as in (1.2) and (1.3),

$$\tilde{\mathcal{B}}(l^2, l^2, \mathbb{T}) \;=\; \mathrm{OCM}(\mathbb{T}). \tag{2.10}$$

Most results in this paper may be regarded as partial results in the direction of conjecture (2.10) (and the same is true for our earlier paper [8]).

In Theorems 2.4–2.6, we give some consequences of Theorem 2.3 which parallel Proposition 3.3, Theorem 1.2 and Proposition 5.1, respectively, of [8]. The proofs are exactly the same as those in [8], provided we replace $\mathfrak{b}$ and $\mathcal{B}$ by $\tilde{\mathfrak{b}}$ and $\tilde{\mathcal{B}}$, respectively.

Theorem 2.4. *Assume* $\mathbb{T}$ *is given by* (1.2), (1.3) *and* B *is an infinite matrix. Then* $B \in \mathrm{OCM}(\mathbb{T})$ *if and only if* B *represents a bounded operator from* l^2 *to* $\tilde{\mathfrak{b}}(l^2, \mathbb{T})$.

Theorem 2.5. *With* $\mathbb{T}$ *and* B *as above, if* $B \in \tilde{\mathcal{B}}(l^2, l^2, \mathbb{T})$ *then* $B \in \mathrm{OCM}(\mathbb{T})$.

Theorem 2.6. *With* $\mathbb{T}$ *and* B *as above, suppose that the rows of* B *have at most* m *nonzero entries* ($m \in \mathbb{N}$). *Then* $B \in \tilde{\mathcal{B}}(l^2, l^2, \mathbb{T})$ *if and only if* $B \in \mathrm{OCM}(\mathbb{T})$.

Remark 2.7. Suppose that the semigroup $\mathbb{T}$ acts on a separable Hilbert space X. Concerning the admissibility for $\mathbb{T}$ of control operators defined on an arbitrary infinite dimensional Hilbert space U, we may restrict our attention, without loss of generality, to the case $U = l^2$ (the approach taken in this paper). Indeed, since X_{-1} is separable, the orthogonal complement of Ker B in U is separable as well (it is the closure of the range of B^*). Restricting B to this subspace then is equivalent to having B defined on l^2. Now consider the case when $X = l^2$ and $\mathbb{T}$ is diagonal. Since any operator from l^2 to l^2_{-1} is represented by an infinite matrix, there is no loss of generality in considering only operators B which are represented by infinite matrices.

Remark 2.8. In the notation of Theorems 2.3 and 2.4, it is *not* true in general that $l^2 \subset \tilde{\mathfrak{b}}(l^2, \mathbb{T})$. In fact if $\mathbb{T}$ is any semigroup on a Hilbert space X and $X \subset \tilde{\mathfrak{b}}(X, \mathbb{T})$, then (2.6) implies that

$$\int_0^\infty |\langle \mathbb{T}_t^* x, b\rangle|^2\, dt < \infty\,, \quad \forall\, x \in X, \quad \forall\, b \in X,$$

i.e., $\mathbb{T}^*$ is weakly L^2-stable. This implies (see Weiss [19]) that $\mathbb{T}^*$ (and hence also $\mathbb{T}$) is exponentially stable. On the other hand it is clear from the definition that $X \subset \mathfrak{b}(X, \mathbb{T})$ regardless of the stability of $\mathbb{T}$.

3. The Lyapunov equation and the controllability Gramian.

In this section we describe the relationship between infinite-time admissibility and the Lyapunov equation. This connection has been investigated by Levan [12] (who assumed that $B \in \mathcal{L}(U, X)$) and by Grabowski [4]. Related results have appeared in Russell and Weiss [13]. The main result of this section is the following theorem, parts of which are contained in [4]. We use the notation of the last section. Recall that a strongly continuous semigroup $\mathbb{T}$ on a Hilbert space X is called *strongly stable* if $\mathbb{T}_t x \to 0$ as $t \to \infty$, for every $x \in X$.

Theorem 3.1. *Let $\mathbb{T}$ be a strongly continuous semigroup on the Hilbert space X, with generator A. Let U be a Hilbert space and assume $B \in \mathcal{L}(U, X_{-1})$. Then the following three statements are equivalent:*

(i) *B is an infinite-time admissible control operator for $\mathbb{T}$.*

(ii) *There exists an operator $P \in \mathcal{L}(X)$ such that for any $x \in Z_1$,*

$$Px = \lim_{\tau \to \infty} \int_0^\tau \mathbb{T}_t BB^* \mathbb{T}_t^* x \, dt \qquad (\text{in } X)\,. \tag{3.1}$$

(iii) *There exist operators $\Pi \in \mathcal{L}(X)$, $\Pi \geq 0$ (possibly only one), which satisfy the following equation with terms in $\mathcal{L}(Z_1, X_{-1})$:*

$$A\,\Pi + \Pi\, A^* = -BB^*. \tag{3.2}$$

Moreover, if B is infinite-time admissible, then the following statements are true:

(I) *P defined in (3.1) is the smallest positive solution of (3.2). In other words $P \geq 0$, P satisfies (3.2) and if $\Pi \in \mathcal{L}(X)$, $\Pi \geq 0$ and (3.2) holds, then $P \leq \Pi$.*

(II) *For any $x \in X$,*

$$\lim_{t \to \infty} P^{\frac{1}{2}} \mathbb{T}_t^* x = 0 \qquad (\text{in } X)\,.$$

In particular, if P is invertible then $\mathbb{T}^*$ is strongly stable.

(III) *If $\mathbb{T}^*$ is strongly stable, then P is the unique self-adjoint solution of (3.2).*

Proof. First we shall prove that (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). As in Section 2, we denote $(\Psi x)(t) = B^* \mathbb{T}_t^* x$, for any $x \in Z_1$ and any $t \geq 0$.

(i) $\Longrightarrow$ (ii): Assume (i) holds. Then $\tilde{\Phi}$ defined in (2.5) is a bounded operator from $L^2([0,\infty), U)$ to X. We define $P = \tilde{\Phi}\tilde{\Phi}^*$, so that $P \in \mathcal{L}(X)$. Then (2.5) and (2.7) show that for any $x \in Z_1$, Px is given by (3.1), so that (ii) holds.

(ii) $\Longrightarrow$ (i): Let $P \in \mathcal{L}(X)$ be defined by (3.1) (this formula defines P since Z_1 is dense in X). Since for any $x \in Z_1$, $\|\Psi x\|^2 = \lim_{\tau \to \infty} \|\Psi x\|^2_{L^2([0,\tau],U)}$, we get

$$\|\Psi x\|^2 = \langle Px, x\rangle\,, \qquad \forall x \in Z_1\,. \tag{3.3}$$

This shows that (2.6) holds (with $k^2 = \|P\|$), so B is infinite-time admissible.

(ii) $\Longrightarrow$ (iii): Let $P \in \mathcal{L}(X)$ be defined by (3.1). We show that (3.2) is satisfied for $\Pi = P$. Let $x, y \in Z_2$ and for $t \geq 0$ define $f(t) = \langle B^* \mathbb{T}_t^* x, B^* \mathbb{T}_t^* y\rangle$. Then f is continuously differentiable and

$$\frac{d}{dt} f(t) = \langle B^* \mathbb{T}_t^* A^* x, B^* \mathbb{T}_t^* y\rangle + \langle B^* \mathbb{T}_t^* x, B^* \mathbb{T}_t^* A^* y\rangle\,.$$

Integrating both sides on $[0, \tau]$ gives

$$f(\tau) - f(0) = \left\langle \int_0^\tau \mathbb{T}_t BB^* \mathbb{T}_t^* A^* x \, dt, y \right\rangle + \left\langle \int_0^\tau \mathbb{T}_t BB^* \mathbb{T}_t^* x \, dt, A^* y \right\rangle. \tag{3.4}$$

Since $A^*x \in Z_1$, by (ii) each of the above integrals converges (in X) as $\tau \to \infty$. Hence $\lim_{\tau\to\infty} f(\tau)$ also exists. Since by (ii) the integral $\int_0^\tau f(t)dt$ has a finite limit as $\tau \to \infty$, we must have $f(\tau) \to 0$ as $\tau \to \infty$. We then let $\tau \to \infty$ in (3.4) to find that

$$\langle PA^*x, y\rangle + \langle APx, y\rangle \;=\; -\langle BB^*x, y\rangle\,.$$

Since Z_2 is dense in Z_1, the above equality holds for any $y \in Z_1$. This implies $(AP + PA^*)x = -BB^*x$, for any $x \in Z_2$. Since both $AP + PA^*$ and BB^* are in $\mathcal{L}(Z_1, X_{-1})$, again by a density argument P satisfies (3.2).

(iii) $\Longrightarrow$ (i): Assume $\Pi \in \mathcal{L}(X)$, $\Pi \geq 0$ and Π satisfies (3.2). For any $x \in X$ and any $t \in [0, \infty)$, we define $E_t(x)$ by $E_t(x) = \langle \Pi\mathbb{T}_t^*x, \mathbb{T}_t^*x\rangle$. Then $E_t(x) \geq 0$ and for any $x \in Z_1$, $E_t(x)$ is a continuously differentiable function of t. By (3.2) we have that for any $x \in Z_1$,

$$\frac{d}{dt}E_t(x) \;=\; -\langle BB^*\mathbb{T}_t^*x, \mathbb{T}_t^*x\rangle \;=\; -\|B^*\mathbb{T}_t^*x\|^2 \;\leq\; 0\,, \tag{3.5}$$

so that $E_t(x)$ is nonincreasing. Since $E_t(x)$ is a continuous function of x, from the density of Z_1 in X we conclude that for any $x \in X$, $E_t(x)$ is nonincreasing. This can be written in the following form: for $0 \leq \tau \leq t$,

$$\mathbb{T}_t\Pi\mathbb{T}_t^* \;\leq\; \mathbb{T}_\tau\Pi\mathbb{T}_\tau^*\,.$$

It is a well known fact that any decreasing positive operator-valued function has a strong limit. Thus, there exists $\Pi_\infty \in \mathcal{L}(X)$, $\Pi_\infty \geq 0$, such that for all $x \in X$

$$\lim_{t\to\infty} \mathbb{T}_t\Pi\mathbb{T}_t^*x \;=\; \Pi_\infty x \qquad (\text{in } X)\,. \tag{3.6}$$

It is clear that $0 \leq \Pi_\infty \leq \Pi$. Integrating (3.5) on $[0, \infty)$ we get that for $x \in Z_1$

$$\langle \Pi x, x\rangle - \langle \Pi_\infty x, x\rangle \;=\; \int_0^\infty \|B^*\mathbb{T}_t^*x\|^2 dt \;=\; \|\Psi x\|^2_{L^2([0,\infty),U)}\,. \tag{3.7}$$

From here we see that (2.6) is satisfied, so that (i) holds.

Now assume that B is infinite-time admissible and let us prove statement (I). We see from (3.3) that $P \geq 0$. We have seen earlier that P satisfies (3.2). If $\Pi \in \mathcal{L}(X)$, $\Pi \geq 0$ and (3.2) holds, then by (3.3) and (3.7) we have that for all $x \in Z_1$

$$\langle Px, x\rangle \;=\; \langle \Pi x, x\rangle - \langle \Pi_\infty x, x\rangle\,. \tag{3.8}$$

By continuity, this remains true for all $x \in X$, so that $P \leq \Pi$, as claimed in (I).

To prove (II), we take $\Pi = P$ in (3.6) and (3.8) and obtain $\Pi_\infty = 0$. By (3.6) this implies $\lim_{t\to\infty}\langle \mathbb{T}_t P\mathbb{T}_t^*x, x\rangle = 0$ for any $x \in X$, which is precisely (II).

Finally, to prove (III) assume that $\mathbb{T}^*$ is strongly stable and $\Pi = \Pi^*$ is a solution of (3.2). Then by the argument in the proof of (iii) $\Longrightarrow$ (i), (3.5) holds. Formula (3.6) holds as well, although for a different reason: it follows from the strong stability of $\mathbb{T}^*$. Moreover, we have $\Pi_\infty = 0$. We obtain (3.7) by integration, which shows that $\langle \Pi x, x\rangle = \langle Px, x\rangle$ for all $x \in X$. Π and P being self-adjoint, we conclude that $\Pi = P$. □

If B is infinite-time admissible for $\mathbb{T}$, then P defined in (3.1) is called the *controllability Gramian* of $\mathbb{T}$ and B. Equation (3.2) is called a *Lyapunov equation* (this name is also used for slightly different equations). In several papers, the connection between the solvability of a Lyapunov equation and the stability of $\mathbb{T}$ was investigated; see, e.g., Levan [12] and the references therein. (In [12] it is claimed that $\mathbb{T}$ uniformly bounded implies the uniqueness of P, which is wrong even if X is one dimensional.)

By (II) of Theorem 3.1, if P is invertible then $\mathbb{T}^*$ is strongly stable. This is the best possible result under the given assumptions, since $\mathbb{T}$ does not have to be strongly stable, see Example 3.4 below.

By (III) of Theorem 3.1, if $\mathbb{T}^*$ is strongly stable then the solution of (3.2) is unique. If $\mathbb{T}$ is strongly stable but $\mathbb{T}^*$ is not, then (3.2) may have many self-adjoint solutions. For example, if $\mathbb{T}$ is the left shift semigroup on $L^2[0,\infty)$ and $B = 0$, then any multiple of the identity I satisfies (3.2).

The following proposition is a slight generalization of a result in [12] (where only the case $B \in \mathcal{L}(U, X)$ is considered).

Proposition 3.2. *With the notation of Theorem* 3.1, *assume that B is an infinite-time admissible control operator for $\mathbb{T}$. If $P > 0$ and $\mathbb{T}$ is uniformly bounded, then $\mathbb{T}$ is weakly stable, i.e., $\langle \mathbb{T}_t x, y\rangle \to 0$ as $t \to \infty$, for any $x, y \in X$.*

Weak stability is the strongest possible conclusion under the assumptions of the proposition, as Example 3.5 shows.

Proof. Denote $V = \operatorname{Ran} P^{\frac{1}{2}}$, then V is dense in X (because $\overline{V}$ is the orthogonal complement of $\operatorname{Ker} P^{\frac{1}{2}} = \operatorname{Ker} P = \{0\}$). It follows from (II) of Theorem 3.1 that for any $x \in X$ and any $v \in V$, $\lim_{t\to\infty}\langle \mathbb{T}_t x, v\rangle = 0$. Let $x, y \in X$ be fixed. We claim that for any $\varepsilon > 0$ we can find $T \geq 0$ such that $\langle \mathbb{T}_t x, y\rangle \leq \varepsilon$ for each $t \geq T$. Indeed, let $v \in V$ be such that $\langle \mathbb{T}_t x, y - v\rangle \leq \frac{\varepsilon}{2}$ for all $t \geq 0$ (this is possible by the uniform boundedness of $\mathbb{T}$). Now if T is such that $\langle \mathbb{T}_t x, v\rangle \leq \frac{\varepsilon}{2}$ for all $t \geq T$, then T is the desired number. The existence of such a T for any $\varepsilon > 0$ means that $\langle \mathbb{T}_t x, y\rangle \to 0$. □

Remark 3.3. After Proposition 3.2 it is worth mentioning the following facts: Suppose that $\mathbb{T}$ is a weakly stable (hence uniformly bounded) semigroup on the Hilbert space X, and let A denote its generator.

(a) If for some (hence for any) $s \in \rho(A)$, $(sI - A)^{-1}$ is compact, then $\mathbb{T}$ and $\mathbb{T}^*$ are strongly stable. This is well known and easy to prove.

(b) If $\sigma(A) \cap i\mathbb{R}$ is at most countable, then $\mathbb{T}$ and $\mathbb{T}^*$ are strongly stable. This follows from the stability theorem of Arendt and Batty [1].

Example 3.4. Let $\mathbb{T}$ be the right shift semigroup on $X = L^2[0,\infty)$. Thus, $Ax = -x'$ for all $x \in D(A)$, where $D(A) = \{x \in H^1[0,\infty), \,|\, x(0) = 0\}$. We take $U = \mathbb{C}$ and $B = \delta_0$, i.e., $B^* x = x(0)$ for each $x \in Z_1 = H^1[0,\infty)$. Then it is not difficult to see that B

is infinite-time admissible and $P = I$. We have that $\mathbb{T}^*$ (the left shift semigroup) is strongly stable, as stated in (II) of Theorem 3.1, but $\mathbb{T}$ is not strongly stable.

Example 3.5. This is a refinement of the preceding example. Let $\mathbb{T}$ be the right shift semigroup on $X = L^2(\mathbb{R})$. Thus, $Ax = -x'$ for all $x \in D(A) = H^1(\mathbb{R})$. We take $U = l^2$ and decompose $Bv = b_1 v_1 + b_2 v_2 + b_3 v_3 ...$, for any $v = (v_1, v_2, v_3 ...) \in l^2$. We define the components of B by $b_k = 2^{-k/2}\delta_{-k}$, i.e., $b_k^* x = 2^{-k/2} x(-k)$ for each $x \in Z_1 = H^1(\mathbb{R})$. Then it is not difficult to see that each b_k is infinite-time admissible and $|||b_k|||_\infty = 2^{(1-k)/2}|||b_1|||_\infty$. Thus, B can be written as the sum of an absolutely convergent series in $\tilde{\mathcal{B}}(U, X, \mathbb{T})$, so that B is infinite-time admissible. A simple computation shows that for any $x \in X$, $(Px)(\xi) = \varphi(\xi)x(\xi)$, where $\varphi(\xi) = \sum_{k \geq -\xi} 2^{-k}$. In particular, $\varphi(\xi) = 1$ for $\xi \in [-1, \infty)$ and $\varphi(\xi)$ decreases rapidly as $\xi \to -\infty$. Since $\varphi(\xi) > 0$ everywhere, we have $P > 0$. By Proposition 3.2 $\mathbb{T}$ is weakly stable, but no stronger stability concepts are true for $\mathbb{T}$, since it is unitary.

Remark 3.6. Assume B is an admissible control operator for $\mathbb{T}$, and let Ψ be as in (2.7). We have seen in Section 2 that for any $x \in Z_1$, $(\Psi x)(t) = B^* \mathbb{T}_t^* x$. In order to obtain a formula valid for any $x \in X$, we may replace B^* by its Λ-*extension* B_Λ^*. We shall say more about this extension in Section 4. We have that for every $x \in X$ and almost every $t \geq 0$,

$$(\Psi x)(t) \;=\; B_\Lambda^* \mathbb{T}_t^* x \,.$$

Together with $P = \tilde{\Phi}\Psi$ this leads to the following expression for the controllability Gramian P (valid for every $x \in X$):

$$Px \;=\; \lim_{\tau \to \infty} \int_0^\tau \mathbb{T}_t \, B B_\Lambda^* \mathbb{T}_t^* x \, dt \qquad (\text{in } X)\,.$$

4. Feedback invariance of admissibility.

In this section we first recall some facts about regular linear systems and especially about feedback for such systems, following Weiss [22], [23], as needed for the proof of our feedback invariance results which are given at the end of this section. We assume that the reader has some familiarity with the concept of regular linear system, as presented in [22], so that our discussion is very brief.

An *abstract linear system* is a linear time-invariant system such that on any finite time interval, the operator from the initial state and the input function to the final state and the output function is bounded. The input, state and output spaces are Hilbert spaces and the input and output functions are of class L^2_{loc}. For the detailed definition we refer to Salamon [15] or to [22]. The input to output operator $\mathbb{F}$ of any abstract linear system can be described by a *transfer function* $\mathbf{G}$, which is an analytic operator-valued function defined on some right half-plane in $\mathbb{C}$. This means that $y = \mathbb{F}u$ iff $\hat{y}(s) = \mathbf{G}(s)\hat{u}(s)$ holds for all s in some right half-plane, where a hat denotes the Laplace transformation.

The transfer function $\mathbf{G}$ is always *well posed*, meaning that it is bounded on some right half-plane. We do not distinguish between two transfer functions defined on two different right half-planes, if one function is a restriction of the other to a smaller half-plane (thus, by a transfer function we mean in fact an equivalence class of analytic functions). We write $\mathbf{G} \in H^\infty$ if $\mathbf{G}$ is bounded on $\mathbb{C}_0$.

Let $\mathbf{\Sigma}$ be an abstract linear system, with input space U, state space X and output space Y. Let $\mathbb{T}$ be the *semigroup of* $\mathbf{\Sigma}$, i.e., the strongly continuous semigroup on X which describes the evolution of the state of $\mathbf{\Sigma}$ if the input function is zero. Let A denote the generator of $\mathbb{T}$. The Hilbert space X_{-1} is defined as in Section 2 and we denote the extension of $\mathbb{T}$ to X_{-1} and the extension of A to X by the same symbols. The state of $\mathbf{\Sigma}$ at any moment $t \geq 0$ can be expressed by the formula (2.3) (see [16], [17]). Here $u \in L^2_{loc}([0,\infty), U)$ is the input function, x_0 is the initial state and $B \in \mathcal{L}(U, X_{-1})$ is the *control operator* of $\mathbf{\Sigma}$. We have $x(t) \in X$ and $x(t)$ depends continuously on t, on x_0 and on the restriction of u to $[0,t]$ (which is in $L^2([0,t],U)$). Thus B is an admissible control operator for $\mathbb{T}$, as defined in Section 2.

The domain $D(A)$ with the norm $\|x\|_1 = \|(\beta I - A)x\|$ becomes a Hilbert space, which we denote by X_1. Here, the choice of $\beta \in \rho(A)$ is irrelevant for the topology of X_1. (This space resembles Z_1 defined in Section 2.) If $u = 0$ and $x_0 \in X_1$, then the output function of $\mathbf{\Sigma}$ on $[0,\infty)$ is (see [16], [20])

$$y(t) \;=\; C\mathbb{T}_t x_0 \,.$$

Here $C \in \mathcal{L}(X_1, Y)$ is the *observation operator* of $\mathbf{\Sigma}$. C is an *admissible observation operator* for $\mathbb{T}$, which means that for some (hence for any) $\tau > 0$, there exists a $k_\tau \geq 0$ such that

$$\int_0^\tau \|C\mathbb{T}_t x_0\|_Y^2 \, dt \;\leq\; k_\tau^2 \|x_0\|_X^2 \,, \qquad \forall\, x_0 \in X_1 \,.$$

It is clear that C is an admissible observation operator for $\mathbb{T}$ if and only if C^* is an admissible control operator for $\mathbb{T}^*$ (see Section 2). The Λ-*extension* of C is defined by

$$C_\Lambda x_0 \;=\; \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1} x_0 \tag{4.1}$$

(λ is real), for all x_0 in the domain

$$D(C_\Lambda) \;=\; \{x_0 \in X \;\big|\; \text{the limit in (4.1) exists}\} \,.$$

Let $\mathbf{G}$ denote the transfer function of $\mathbf{\Sigma}$. $\mathbf{G}$ is called *regular* if the following limit exists for all $\mathrm{v} \in U$:

$$D\mathrm{v} \;=\; \lim_{\lambda \to +\infty} \mathbf{G}(\lambda)\mathrm{v} \tag{4.2}$$

(λ is real). Then $D \in \mathcal{L}(U, Y)$ is called the *feedthrough operator* of $\mathbf{G}$. If $\mathbf{G}$ is regular then $\mathbf{\Sigma}$ is called a *regular linear system*, abreviated RLS. The regularity condition (4.2) can be formulated in many different ways, of which we mention the following: $\mathbf{\Sigma}$ is regular if and only if the product $C_\Lambda(sI - A)^{-1}B$ makes sense for some (hence for any) $s \in \rho(A)$, i.e.,

$$\mathrm{Ran}\,(sI - A)^{-1}B \;\subset\; D(C_\Lambda)$$

(see [22]). If $\mathbf{\Sigma}$ is regular, then

$$\mathbf{G}(s) = C_\Lambda (sI - A)^{-1} B + D \,, \tag{4.3}$$

for any $s \in \mathbb{C}$ with $\mathrm{Re}\, s$ sufficiently large. Moreover, in the time domain, $\mathbf{\Sigma}$ is completely described by the following equations:

$$\dot{x}(t) = A\, x(t) + B\, u(t) \,, \tag{4.4a}$$

$$y(t) = C_\Lambda x(t) + D\, u(t) \,, \tag{4.4b}$$

which hold for almost every $t \geq 0$ (in particular, $x(t) \in D(C_\Lambda)$ for a.e. $t \geq 0$). For any given $x_0 \in X$ and $u \in L^2_{loc}([0,\infty), U)$, $x(\cdot)$ is the unique strong solution (in X_{-1}) of (4.4a). The output function y belongs to $L^2_{loc}([0,\infty), Y)$. The operators A, B, C and D are called the *generating operators* of $\mathbf{\Sigma}$, or its *generators* for short.

Let U, X and Y be Hilbert spaces and A, B and C linear operators. We say that (A, B, C) is a *regular triple* on U, X and Y if for some (hence for any) $D \in \mathcal{L}(U, Y)$, A, B, C, D are the generators of an RLS with input, state and output spaces U, X and Y. We have the following:

Proposition 4.1. *(A, B, C) is a regular triple on U, X and Y if and only if the following five conditions are satisfied:*

1. *A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X.*
2. *$B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$.*
3. *$C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$.*
4. *$C_\Lambda (sI - A)^{-1} B$ makes sense for some (hence for any) $s \in \rho(A)$.*
5. *$s \to \|C_\Lambda (sI - A)^{-1} B\|_{\mathcal{L}(U,Y)}$ is bounded on some right half-plane.*

Now we turn to feedback, following [23]. Let U and Y be Hilbert spaces, suppose $\mathbf{G}$ is an $\mathcal{L}(U, Y)$-valued well posed transfer function and let $K \in \mathcal{L}(Y, U)$. K is an *admissible feedback operator* for $\mathbf{G}$ if $I - K\mathbf{G}$ is invertible on some right half-plane and its inverse is a well posed transfer function (equivalently, if $I - \mathbf{G}K$ has the same property). Then the function $\mathbf{G}^K$ defined by

$$\mathbf{G}^K(s) = \mathbf{G}(s)(I - K\mathbf{G}(s))^{-1} \tag{4.5}$$

is called the *closed loop transfer function* corresponding to $\mathbf{G}$ and K. If $\mathbf{G}$ is regular and its feedthrough operator is zero, then $\mathbf{G}^K$ (given by (4.5)) is again regular and its feedthrough operator is also zero.

Let $\mathbf{\Sigma}$ be an abstract linear system with transfer function $\mathbf{G}$. If K is an admissible feedback operator for $\mathbf{G}$ then there exists a unique *closed loop system* $\mathbf{\Sigma}^K$ corresponding to $\mathbf{\Sigma}$ and K, represented in Figure 1. $\mathbf{\Sigma}^K$ is an abstract linear system and its transfer function is $\mathbf{G}^K$ from (4.5). For the precise definition of $\mathbf{\Sigma}^K$ we refer to [23].

Now suppose that $\mathbf{\Sigma}$ is an RLS with generators A, B, C, 0 and suppose that K and $\mathbf{\Sigma}^K$ are as before. Then (as mentioned earlier) $\mathbf{\Sigma}^K$ is regular and we can compute its generators via formulas which are similar to those for finite dimensional systems.

Let A^K, B^K, C^K, D^K be the generators of $\mathbf{\Sigma}^K$. As we already know, $D^K = 0$. The formula for A^K is

$$A^K x = (A + BKC_\Lambda) x , \tag{4.6}$$

defined for all x in the domain

$$D(A^K) = \{ x \in D(C_\Lambda) \mid (A + BKC_\Lambda) x \in X \} .$$

We call $D(A^K)$ defined above the *natural domain* of $A + BKC_\Lambda$. We have

$$C^K x = C_\Lambda x , \qquad \forall\, x \in D(A^K) , \tag{4.7}$$

while $B^K = B$. To understand the last formula better, it should be pointed out that $B^K \in \mathcal{L}(U, X_{-1}^K)$, where X_{-1}^K is the analogue of X_{-1} for $\mathbf{\Sigma}^K$, but in fact both B and B^K are in $\mathcal{L}(U, W)$, where W is a Banach space such that $W \subset X_{-1} \cap X_{-1}^K$. W depends only on A and C, X is dense in W and $C_\Lambda (sI - A)^{-1}$ has a continuous extension to W.

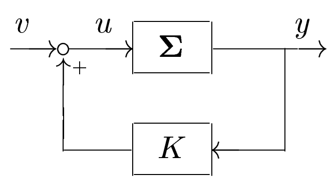

Fig. 1. The closed loop system Σ^K

The extensions C_Λ and C^K_Λ (defined as in (4.1)) are equal: $C^K_\Lambda = C_\Lambda$.
The resolvents of A and A^K are related by

$$\begin{aligned}(sI - A^K)^{-1} - (sI - A)^{-1} &= (sI - A)^{-1} BKC_\Lambda (sI - A^K)^{-1} \\ &= (sI - A^K)^{-1} BKC (sI - A)^{-1} ,\end{aligned}$$

for any $s \in \mathbb{C}$ with Re s sufficiently large. Let Φ_τ ($\tau \geq 0$) be the input to state operators of $\mathbf{\Sigma}$, as in Definition 2.1, and let Φ_τ^K be the corresponding operators for $\mathbf{\Sigma}^K$. Let $\mathbb{F}^K$ denote the input to output operator of $\mathbf{\Sigma}^K$ (corresponding to the transfer function $\mathbf{G}^K$). Then

$$\Phi_\tau^K = \Phi_\tau (I + K\mathbb{F}^K) . \tag{4.8}$$

Proposition 4.2. *Let $\mathbf{\Sigma}$ be an* RLS *with generators A, B, C, 0 and transfer function $\mathbf{G}$. Let K be an admissible feedback operator for $\mathbf{G}$. Let $\mathbb{T}^K$ be the semigroup of the corresponding closed loop system. Then B is an admissible control operator for $\mathbb{T}^K$.*

Let $\mathbf{G}^K$ be the closed loop transfer function from (4.5). *If B is infinite-time admissible for $\mathbb{T}$ and $K\mathbf{G}^K \in H^\infty$, then B is infinite-time admissible for $\mathbb{T}^K$.*

Proof. Let $\mathbf{\Sigma}^K$ denote the closed loop system. Since B is the control operator of $\mathbf{\Sigma}^K$ and $\mathbb{T}^K$ is its semigroup, it is clear that B is an admissible control operator for $\mathbb{T}^K$.

The second part follows from (4.8) and from the fact that $I + K\mathbb{F}^K$ is a bounded operator on $L^2([0, \infty), U)$, so that the operators Φ_τ^K are uniformly bounded. $\square$

Notation. We shall use the particular case of Proposition 4.2 corresponding to the following system. The state space $X = l^2$ and $\mathbb{T}$ is diagonal, given by (1.2) and (1.3). Thus, $A = \operatorname{diag}(\lambda_1, \lambda_2, ...)$, where $-\lambda_k \in \mathbb{C}_0$. The input space U and the control operator B are decomposed as follows:

$$B = [\, B_1 \quad B_2 \,] \,, \qquad U = \begin{matrix} l^2 \\ \times \\ l^2 \end{matrix} .$$

We take $Y = l^2$ and we assume that B_2 and C are represented by infinite diagonal matrices,

$$B_2 = \operatorname{diag}(\beta_1, \beta_2, ...) \,, \qquad C = \operatorname{diag}(c_1, c_2, ...) \,. \tag{4.9}$$

The sequences (β_k) and (c_k) are such that $\beta_k \geq 0$ and

$$\beta_k^2 = \frac{1}{2} |\mathrm{Re}\ \lambda_k| \,, \qquad |c_k|^2 \leq \frac{1}{2} |\mathrm{Re}\ \lambda_k| \,. \tag{4.10}$$

Thus, β_k is determined by λ_k. Note that by (4.10), $B_2 \in \mathrm{OCM}(\mathbb{T})$ and $C^* \in \mathrm{OCM}(\mathbb{T}^*)$.

We assume that the infinite matrix B_1 represents an admissible control operator for $\mathbb{T}$, i.e., $B_1 \in \mathcal{B}(l^2, l^2, \mathbb{T})$ and we shall not distinguish between the matrix and this operator. We denote by b_k the k-th row of B_1, so that $b_k \in l^{2*}$.

It is clear that the numbers $-(\lambda_k + \beta_k c_k)$ are in $\mathbb{C}_0$. We define a new semigroup $\mathbb{T}^+$ on l^2 by

$$(\mathbb{T}_t^+ z)_k = e^{(\lambda_k + \beta_k c_k)t} z_k \,, \qquad \forall\, k \in \mathbb{N}, \tag{4.11}$$

so that $\mathbb{T}^+$ is generated by $A^+ = \operatorname{diag}(\lambda_1 + \beta_1 c_1, \lambda_2 + \beta_2 c_2, ...)$. Note that the space l^{2+}_{-1} for this semigroup is the same as l^2_{-1} for the original semigroup $\mathbb{T}$.

Proposition 4.3. *With the above notation, if B_1 and C are such that (A, B_1, C) is a regular triple, then B_1 is an admissible control operator for $\mathbb{T}^+$.*

If moreover B_1 is infinite-time admissible for $\mathbb{T}$ and the function $s \to C_\Lambda (sI - A)^{-1} B_1$ is in H^∞, then B_1 is infinite-time admissible for $\mathbb{T}^+$.

Proof. First we show that (A, B, C) is a regular triple. This is equivalent to the statement that both (A, B_1, C) and (A, B_2, C) are regular triples. For (A, B_1, C) this has been assumed in the proposition, while for (A, B_2, C) we must check the conditions listed in Proposition 4.1. The first condition is clearly satisfied. Since $B_2 \in \mathrm{OCM}(\mathbb{T})$ and $C^* \in \mathrm{OCM}(\mathbb{T}^*)$, by Theorem 2.6 (with $m = 1$) B_2 is an admissible control operator for $\mathbb{T}$ and C is an admissible observation operator for $\mathbb{T}$. Moreover, it is easy to check that the composition $C_\Lambda (sI - A)^{-1} B_2$ makes sense for every $s \in \mathbb{C}_0$. The function $s \to \|C_\Lambda (sI - A)^{-1} B_2\|$ is bounded on $\mathbb{C}_0$, since for all $s \in \mathbb{C}_0$

$$\left\|C_\Lambda (sI - A)^{-1} B_2\right\|_{\mathcal{L}(l^2)} = \left\|\operatorname{diag}\left(\frac{c_1 \beta_1}{s - \lambda_1}, \frac{c_2 \beta_2}{s - \lambda_2}, ...\right)\right\|_{\mathcal{L}(l^2)}$$
$$\leq \sup_{k \in \mathbb{N}} \left|\frac{c_k \beta_k}{\mathrm{Re}\ \lambda_k}\right| \leq \frac{1}{2} .$$

Thus we conclude that (A, B, C) is a regular triple.

We define $\mathbf{\Sigma}$ to be the RLS with generators A, B, C and 0. By (4.3) its transfer function is $\mathbf{G}(s) = [\, C_\Lambda (sI - A)^{-1} B_1 \quad C_\Lambda (sI - A)^{-1} B_2 \,]$. We take $K = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and claim that K is an admissible feedback operator for $\mathbf{G}$. Indeed, for every $s \in \mathbb{C}_0$

$$\|\mathbf{G}(s)K\|_{\mathcal{L}(Y)} \;=\; \left\|C_\Lambda (sI - A)^{-1} B_2\right\|_{\mathcal{L}(l^2)} \;\leq\; \frac{1}{2}\,,$$

so that $I - \mathbf{G}(s)K$ is invertible on $\mathbb{C}_0$ and its inverse is bounded. We conclude that there is a closed loop system $\mathbf{\Sigma}^K$ corresponding to $\mathbf{\Sigma}$ and K and let $\mathbb{T}^K$ denote its semigroup. Then from (4.6) we get that the generator of $\mathbb{T}^K$ is

$$A^K \;=\; A^+ \;=\; \operatorname{diag}(\lambda_1 + \beta_1 c_1, \lambda_2 + \beta_2 c_2, \ldots)\,,$$

with its natural domain, so that $\mathbb{T}^K = \mathbb{T}^+$. By the first part of Proposition 4.2 we have that B is admissible for $\mathbb{T}^+$, in particular B_1 is admissible for $\mathbb{T}^+$.

In order to prove the second part of the proposition, we note that B_2 is in fact infinite-time admissible (by Theorem 2.6), so that if B_1 has this property too, then B is infinite-time admissible for $\mathbb{T}$. Assume that $s \to C_\Lambda (sI - A)^{-1} B_1$ is in H^∞. Since $s \to C_\Lambda (sI - A)^{-1} B_2$ is in H^∞ too (its supremum is $\leq 1/2$), it follows that $\mathbf{G} \in H^\infty$. Since $\mathbf{G}^K = (I - \mathbf{G}K)^{-1}\mathbf{G}$ and $(I - \mathbf{G}K)^{-1} \in H^\infty$, it follows that $\mathbf{G}^K \in H^\infty$. Thus, all the conditions are satisfied for using the second part of Proposition 4.2 and we conclude that B (in particular, B_1) is infinite-time admissible for $\mathbb{T}^+$. $\square$

In order to apply the last proposition, we must be able to determine when the triple (A, B_1, C) is regular, and if it is, we must be able to compute $C_\Lambda (sI - A)^{-1} B_1$. All this should be possible in terms of the sequences (λ_k), (b_k) and (c_k). The following proposition is meant to be helpful in this regard.

Proposition 4.4. *With the notation introduced before Proposition* 4.3*, suppose that for some* $\delta \geq 0$ *and every* $v \in l^2$

$$\sup_{\operatorname{Re} s = \delta} \sum_{k \in \mathbb{N}} \left| \frac{c_k b_k v}{s - \lambda_k} \right|^2 \;<\; \infty\,. \tag{4.12}$$

Then (A, B_1, C) *is a regular triple and for every* $s \in \mathbb{C}_\delta$*, the function* $\mathbf{G}_1(s) = C_\Lambda (sI - A)^{-1} B_1$ *is given componentwise by*

$$(\mathbf{G}_1(s)v)_k \;=\; \frac{c_k b_k v}{s - \lambda_k}\,. \tag{4.13}$$

Proof. Suppose that (4.12) holds. It is clear that the expression on the left-hand side of (4.12) is nonincreasing as a function of δ, so that

$$\sup_{s \in \mathbb{C}_\delta} \sum_{k \in \mathbb{N}} \left| \frac{c_k b_k v}{s - \lambda_k} \right|^2 \;<\; \infty\,. \tag{4.14}$$

For every $s \in \mathbb{C}_\delta$, let $\mathbf{G}_1(s)$ be the operator from l^2 to l^2 defined in (4.13). It is easily seen that $\mathbf{G}_1(s)$ is closed, and hence bounded (by the closed graph theorem), i.e.,

$\mathbf{G}_1(s) \in \mathcal{L}(l^2)$. Now (4.14) can be written in the form $\sup_{s\in\mathbb{C}_\delta} \|\mathbf{G}_1(s)v\| < \infty$. By the uniform boundedness theorem it follows that

$$\sup_{s\in\mathbb{C}_\delta} \|\mathbf{G}_1(s)\| < \infty . \tag{4.15}$$

Further, it is easy to show (using (4.13)) that for every $v \in l^2$

$$\lim_{\lambda\to+\infty} \mathbf{G}_1(\lambda)\, v = 0 . \tag{4.16}$$

For every $s, \lambda \in \mathbb{C}_\delta$ and $v \in l^2$ we have

$$\left[\frac{\mathbf{G}_1(s) - \mathbf{G}_1(\lambda)}{s-\lambda} v\right]_k = -\frac{c_k b_k v}{(\lambda-\lambda_k)(s-\lambda_k)} . \tag{4.17}$$

Since $\frac{b_k v}{s-\lambda_k}$ are the terms of the l^2 sequence $(sI-A)^{-1}B_1 v$, it follows that

$$\begin{aligned} \mathbf{G}_1(s)\, v - \mathbf{G}_1(\lambda)\, v &= (\lambda - s) C(\lambda I - A)^{-1}(sI-A)^{-1}B_1 v \\ &= \frac{\lambda - s}{\lambda} C\lambda(\lambda I - A)^{-1}(sI-A)^{-1}B_1 v . \end{aligned}$$

According to (4.16), the left-hand side has a limit as $\lambda \to +\infty$, namely $\mathbf{G}_1(s)v$. By the definition of C_Λ, it follows that $(sI-A)^{-1}B_1 v \in D(C_\Lambda)$ and

$$\mathbf{G}_1(s)\, v = C_\Lambda (sI-A)^{-1}B_1 v .$$

Thus, (A, B_1, C) is regular, since the product $C_\Lambda(sI-A)^{-1}B_1$ makes sense and (according to (4.15)) it is bounded on a right half-plane (see Proposition 4.1). □

Remark 4.5. The converse of Proposition 4.4 is also true, i.e., if (A, B_1, C) is a regular triple, then for some $\delta \geq 0$ and every $v \in l^2$, the inequality (4.12) holds (and hence also the representation (4.13)). To prove this, we can use (4.17), in which we multiply both sides with λ and take limits as $\lambda \to +\infty$.

Remark 4.6. The previous remark enables us to construct examples of operators A, B_1 and C satisfying all the assumptions listed in the Notation segment before Proposition 4.3, and such that (A, B_1, C) is not a regular triple. For example, we may take $A = -2\,\mathrm{diag}\,(1^2, 2^2, 3^2, ...)$, $(B_1 v)^* = [\sqrt{1}\ \sqrt{2}\ \sqrt{3}\ ...]$, $C = \mathrm{diag}\,(1, 2, 3, ...)$, so that

$$\frac{c_k b_k v}{s - \lambda_k} = \frac{k\sqrt{k}}{s + 2k^2} .$$

The above sequence is not in l^2, so that (4.12) is false.

We introduce a set $S \subset \mathbb{C}_0$ which depends on three parameters a, b and α, with $0 < a \leq b$ and with $0 \leq \alpha < 1$:

$$S = \{z \in \mathbb{C}_0 \mid a|\mathrm{Im}\, z|^\alpha \leq \mathrm{Re}\, z \leq b|\mathrm{Im}\, z|^\alpha\} . \tag{4.18}$$

Note that for $\alpha = 0$ this is a vertical strip. The following theorem is needed for the proof of part (iii) of Theorem 1.5 (in Section 6).

Theorem 4.7. *With the notation introduced before Proposition* 4.3, *suppose that* $\{-\lambda_k \mid k \in \mathbb{N}\} \subset S$, *where* S *is the set from* (4.18).

Then (A, B_1, C) *is a regular triple.*

Proof. We partition S into three sets denoted S_-, S_0 and S_+, according to $\operatorname{Im} z \leq -1$, $|\operatorname{Im} z| < 1$ and $\operatorname{Im} z \geq 1$, respectively. We take $v \in l^2$. For each $s \in \mathbb{C}_0$ we denote

$$h_+(s) \;=\; \sum_{-\lambda_k \in S_+} \left| \frac{c_k b_k v}{s - \lambda_k} \right|^2$$

(we do not claim convergence at this stage), while $h_0(s)$ and $h_-(s)$ are defined similarly, with S_+ replaced by S_0 and by S_-. According to Proposition 4.4, we have to show that $h_-(s) + h_0(s) + h_+(s)$ is bounded on some vertical line $\operatorname{Re} s = \delta$, with $\delta \geq 0$.

First we prove the boundedness of $h_0(s)$. Let $g > 0$ be such that $S_0 \subset R(g, 0)$ (we have used the notation from (1.4)). Since $B_1 v \in \mathrm{b}(l^2, \mathbb{T})$, by Theorem 2.3 we have

$$\sum_{-\lambda_k \in S_0} |b_k v|^2 \;\leq\; Mg$$

and by (4.10) we have $\sup_{-\lambda_k \in S_0} |c_k|^2 \leq \frac{g}{2}$. Hence

$$h_0(s) \;\leq\; \frac{g}{2(\operatorname{Re} s)^2} \sum_{-\lambda_k \in S_0} |b_k v|^2 \;\leq\; \frac{Mg^2}{2(\operatorname{Re} s)^2}\,.$$

Thus, $h_0(s)$ is bounded on the half-plane $\operatorname{Re} s \geq \delta$, for every $\delta > 0$.

It remains to show that $h_-(s)$ and $h_+(s)$ are bounded on some vertical line. We only show that

$$\sup_{\xi \in \mathbb{R}} h_+(-i\xi) \;<\; \infty\,, \tag{4.19}$$

since the proof for $h_-(s)$ is similar. From (4.19) it then follows that $h_+(s)$ is bounded on the half-plane $\operatorname{Re} s \geq 0$, see (4.14).

We define the sequences (x_n) and (y_n) inductively, by $x_0 = b$, $y_0 = 1$, $y_n = y_{n-1} + x_{n-1}$, $x_n = b y_n^\alpha$. Note that the points $x_n + i y_n$ lie on the right boundary of S_+ and $x_n \to \infty$, $y_n \to \infty$ (we advise the reader to draw the picture). We partition S_+ into a sequence (Q_n) of sets defined by

$$Q_n \;=\; \{z \in S_+ \mid y_{n-1} \leq \operatorname{Im} z < y_n\}\,.$$

Since $Q_n \subset R(x_n, y_n)$, we have by Theorem 2.3

$$\sum_{-\lambda_k \in Q_n} |b_k v|^2 \;\leq\; M x_n\,.$$

In the sequel, $m_1, m_2, m_3, \ldots$ will denote positive constants depending only on the parameters a, b and α which determine S. By simple geometric considerations

$$\sup_{-\lambda_k \in Q_n} \frac{1}{|i\xi + \lambda_k|^2} \;\leq\; m_1 \frac{1}{m_2 x_n^2 + (y_n - \xi)^2}$$

and (by (4.10))

$$\sup_{-\lambda_k \in Q_n} |c_k|^2 \le \frac{1}{2} x_n \, .$$

Therefore

$$\begin{aligned} h_+(-i\xi) &= \sum_{n\in\mathbb{N}} \sum_{-\lambda_k\in Q_n} \left| \frac{c_k b_k v}{i\xi + \lambda_k} \right|^2 \\ &\le \sum_{n\in\mathbb{N}} \frac{m_1 x_n}{2\,[m_2 x_n^2 + (y_n - \xi)^2]} \sum_{-\lambda_k\in Q_n} |b_k v|^2 \\ &\le \frac{M m_1}{2} \sum_{n\in\mathbb{N}} \frac{x_n^2}{m_2 x_n^2 + (y_n-\xi)^2} \, . \end{aligned}$$

Since $x_n^2 \le m_3 \lambda(Q_n)$, where λ denotes the Lebesgue measure (area) in $\mathbb{C}$, and since

$$\frac{1}{m_2 x_n^2 + (y_n-\xi)^2} \le m_4 \inf_{x+iy\in Q_n} \frac{1}{m_2 x^2 + (y-\xi)^2} \, ,$$

we have that

$$\frac{x_n^2}{m_2 x_n^2 + (y_n-\xi)^2} \le m_3 m_4 \int\!\!\int_{Q_n} \frac{dx dy}{m_2 x^2 + (y-\xi)^2} \, .$$

Combining this with our previous estimate for $h_+(-i\xi)$, we get

$$\begin{aligned} h_+(-i\xi) &\le M m_5 \int\!\!\int_{S_+} \frac{dx dy}{m_2 x^2 + (y-\xi)^2} \\ &= M m_5 \int_1^\infty dy \int_{a y^\alpha}^{b y^\alpha} \frac{dx}{m_2 x^2 + (y-\xi)^2} \\ &\le M m_5 \int_1^\infty \frac{(b-a) y^\alpha dy}{m_2 a^2 y^{2\alpha} + (y-\xi)^2} \\ &= M m_6 \int_1^\infty \frac{y^\alpha dy}{m_7 y^{2\alpha} + (y-\xi)^2} \, . \end{aligned}$$

The last integral is clearly convergent, so now at least we know that $h_+(-i\xi) < \infty$.

In order to prove the uniform boundedness of $h_+(-i\xi)$, we argue as follows: the last integral, which we denote by $I(\xi)$, is a continuous function of ξ, so that the only possible way for it to be unbounded is to converge to ∞ as $\xi \to -\infty$ or as $\xi \to +\infty$. But for $\xi \to -\infty$ we have $I(\xi) \to 0$, so that we only have to make sure that $I(\xi)$ stays bounded as $\xi \to +\infty$. Thus, in order to prove (4.19) (and hence the theorem), we only have to prove that

$$\limsup_{\xi\to+\infty} \int_1^\infty \frac{y^\alpha dy}{m_7 y^{2\alpha} + (y-\xi)^2} < \infty \, . \tag{4.20}$$

In order to prove (4.20), we decompose $I(\xi)$ into three parts and estimate each separately:

$$I(\xi) \le \int_1^{\xi/2} \frac{y^\alpha dy}{(y-\xi)^2} + \int_{\xi/2}^{2\xi} \frac{y^\alpha dy}{m_7 \left(\frac{\xi}{2}\right)^{2\alpha} + (y-\xi)^2} + \int_{2\xi}^\infty \frac{y^\alpha dy}{m_7 (2\xi)^{2\alpha} + \left(\frac{y}{2}\right)^2}$$

$$\leq \left(\frac{\xi}{2}\right)^{\alpha} \int_0^{\xi/2} \frac{dy}{(y-\xi)^2} + (2\xi)^{\alpha} \int_{-\infty}^{\infty} \frac{dy}{m_7 \left(\frac{\xi}{2}\right)^{2\alpha} + y^2} + 4 \int_0^{\infty} \frac{y^{\alpha} dy}{4m_7 (2\xi)^{2\alpha} + y^2} .$$

Each of the terms above has a finite limit as $\xi \to +\infty$, so that (4.20) is true. □

5. Results for diagonal semigroups.

In the remainder of this paper we shall restrict our attention to the case in which $\mathbb{T}$ is given by (1.2), (1.3) and B is an infinite matrix with rows $b_k \in l^{2*}$.

The following result contains a restatement of (i) of Theorem 1.5 and is analogous to Theorem 3.1 for the case of diagonal semigroups. Perhaps it is of interest apart from control theory, since it relates the boundedness of an infinite matrix which can be written as a controllability Gramian to infinite-time admissibility.

Proposition 5.1. *Let the semigroup $\mathbb{T}$ be defined by (1.2), (1.3) and assume that B is an infinite matrix with rows $b_k \in l^{2*}$. B^* denotes the conjugate transpose matrix of B. Then the following statements are equivalent:*

(i) *B is an infinite-time admissible control operator for $\mathbb{T}$, i.e., $B \in \tilde{\mathcal{B}}(l^2, l^2, \mathbb{T})$.*

(ii) *The infinite matrix with entries*

$$p_{jk} = -\frac{\langle b_j, b_k \rangle}{\lambda_j + \overline{\lambda}_k} \tag{5.1}$$

represents a bounded operator P on l^2.

(iii) *The equation (3.2) has a positive solution in $\mathcal{L}(l^2)$.*

Moreover, if (ii) holds then P defined there is the unique self-adjoint solution of (3.2).

Proof. Note that if we would have assumed that $B \in \mathcal{L}(l^2, l^2_{-1})$, then this proposition would have followed very easily from Theorem 3.1.

First we prove that (i) ⇔ (ii). If B is infinite-time admissible, then clearly $B \in \mathcal{L}(l^2, l^2_{-1})$. By (i) $\Longrightarrow$ (ii) of Theorem 3.1 there is a positive operator $P \in \mathcal{L}(l^2)$ satisfying (3.1). An easy calculation shows that the components of the infinite matrix representing P are given by (5.1). Thus, (p_{jk}) defines a bounded operator on l^2.

Conversely, let $\mathcal{F}$ denote the space of complex sequences with at most finitely many nonzero entries. Then B^* maps $\mathcal{F}$ into l^2. Let P be the bounded operator on l^2 represented by the matrix (p_{jk}). For $x \in \mathcal{F}$ it is not hard to check that $\mathbb{T}_t^* x \in \mathcal{F}$ and

$$\int_0^{\infty} \|B^* \mathbb{T}_t^* x\|^2 \, dt = \langle Px, x \rangle \leq \|P\| \cdot \|x\|^2 . \tag{5.2}$$

This implies that for any $s \in \mathbb{C}_0$ and any $x \in \mathcal{F}$

$$\left\| B^*(sI - A^*)^{-1} x \right\|^2 = \left\| \int_0^{\infty} e^{-st} B^* \mathbb{T}_t^* x dt \right\|^2 \leq \frac{\|P\|}{2 \operatorname{Re} s} \|x\|^2 .$$

Since $\mathcal{F}$ is dense in l^2, this shows that the operator $B^*(sI - A^*)^{-1}$ from $\mathcal{F}$ to l^2 can be extended to an operator in $\mathcal{L}(l^2)$. Since $(sI - A^*)^{-1}$ is an isomorphism from l^2

to Z_1 (as defined in Section 2), it follows that $B^* \in \mathcal{L}(Z_1, l^2)$, which is equivalent to $B \in \mathcal{L}(l^2, l^2_{-1})$. This fact, together with the density of $\mathcal{F}$ in Z_1 implies that (5.2) remains true for any $x \in Z_1$. Thus (2.6) holds, which means that B represents an infinite-time admissible control operator for $\mathbb{T}$.

Let us prove (i) $\Leftrightarrow$ (iii). If B is infinite-time admissible then (iii) follows immediately by Theorem 3.1. Conversely, if (3.2) has a bounded solution then it follows that BB^* represents a bounded operator from Z_1 to l^2_{-1}. This easily implies that $B \in \mathcal{L}(l^2, l^2_{-1})$, and now (i) follows by Theorem 3.1.

Finally, if we assume (i) then P is a positive solution of (3.2), according to (I) of Theorem 3.1. Since $\mathbb{T}^*$ is strongly stable, P is the unique self-adjoint solution of (3.2), by statement (III) of Theorem 3.1. □

Remark 5.2. Consider the following particular case of Proposition 5.1: $\lambda_k = -k$, so that $\mathbb{T}$ is analytic and positive. B has only one nonzero column and it consists only of ones. By Theorem 2.3, B is an infinite-time admissible control operator for $\mathbb{T}$. We have $\langle b_j, b_k \rangle = 1$ for all j, k, so that P is represented by the famous Hilbert matrix $p_{jk} = \frac{1}{j+k}$. As is well known, P is bounded on l^2, and we have found a new proof of this fact.

The following is a reformulation of (ii) of Theorem 1.5.

Proposition 5.3. *Let the semigroup* $\mathbb{T}$ *be defined by* (1.2), (1.3), *let* A *be its generator and assume* B *is an infinite matrix. Then* $B \in \mathrm{OCM}(\mathbb{T})$ *if and only if* $B \in \mathcal{L}(l^2, l^2_{-1})$ *and there exists a* $K \geq 0$ *such that*

$$\|(sI - A)^{-1} B\|^2_{\mathcal{L}(l^2)} \leq \frac{K}{\operatorname{Re} s}, \qquad \forall\, s \in \mathbb{C}_0. \tag{5.3}$$

Proof. Assume first that $B \in \mathcal{L}(l^2, l^2_{-1})$ and (5.3) holds. Then the rows b_k of B must be in l^{2*}. Let $s = h - i\omega \in \mathbb{C}_0$. We observe that

$$\begin{aligned}
\Bigg\| \sum_{-\lambda_k \in R(h,\omega)} b_k^* b_k \Bigg\|_{\mathcal{L}(l^2)} &= \sup_{\|\mathrm{v}\| \leq 1} \langle \sum_{-\lambda_k \in R(h,\omega)} b_k^* b_k \mathrm{v}, \mathrm{v} \rangle \\
&= \sup_{\|\mathrm{v}\| \leq 1} \sum_{-\lambda_k \in R(h,\omega)} \langle b_k^* b_k \mathrm{v}, \mathrm{v} \rangle \\
&= \sup_{\|\mathrm{v}\| \leq 1} \sum_{-\lambda_k \in R(h,\omega)} |b_k \mathrm{v}|^2.
\end{aligned}$$

Since for $-\lambda_k \in R(h, \omega)$ we have $|s - \lambda_k|^2 \leq 5h^2$, we can write

$$\begin{aligned}
\sum_{-\lambda_k \in R(h,\omega)} |b_k \mathrm{v}|^2 &\leq \sum_{-\lambda_k \in R(h,\omega)} \frac{|b_k \mathrm{v}|^2}{|s - \lambda_k|^2} \cdot 5h^2 \\
&\leq 5h^2 \sum_{k=1}^{\infty} \left| \frac{b_k \mathrm{v}}{s - \lambda_k} \right|^2 \\
&= 5h^2 \left\| (sI - A)^{-1} B \mathrm{v} \right\|^2_{l^2} .
\end{aligned}$$

Now (5.3) implies that (1.5) holds with $M = 5K$.

Conversely, assume that $B \in \mathrm{OCM}(\mathbb{T})$. By Theorem 2.4, $B \in \mathcal{L}(l^2, \tilde{\mathfrak{b}}(l^2, \mathbb{T}))$ (in particular, $B \in \mathcal{L}(l^2, l^2_{-1})$). For any $\mathrm{v} \in l^2$ and any $f \in L^2[0, \infty)$ with $\|f\| \leq 1$, we have

$$\left\| \int_0^\infty \mathbb{T}_t B \mathrm{v} f(t)\, dt \right\|_{l^2} \leq |||B\mathrm{v}|||_\infty \leq m \|\mathrm{v}\|_{l^2}\,, \tag{5.4}$$

where m is independent of v. Now let $s \in \mathbb{C}_0$ and use (5.4) with $f(t) = e^{-st}\sqrt{2\mathrm{Re}\ s}$ to obtain

$$\sqrt{2\mathrm{Re}\, s}\, \|(sI - A)^{-1} B \mathrm{v}\|_{l^2} \leq m \|\mathrm{v}\|_{l^2}\,,$$

which implies (5.3) (with $K = m^2/2$). □

6. Extensions of Theorem 1.3.

As we have mentioned, for *arbitrary* systems of the form (1.1)–(1.3) we do not know if infinite-time admissibility is completely determined by the operator Carleson measure criterion, i.e., if (2.10) holds. However, as Theorems 1.2 and 1.3 indicate, for many semigroups $\mathbb{T}$ we know this to be the case. In this final section we show that (2.10) holds for a much larger class of semigroups than those in Theorems 1.2 and 1.3. Throughout this section we assume $\mathbb{T}$ and B to be as in Theorem 1.5.

We first introduce some notation which we shall use to reformulate the operator Carleson measure criterion. For any $h > 0$ and $\omega \in \mathbb{R}$ let

$$Q(h, \omega) = \left\{ z \in \mathbb{C} \;\middle|\; \frac{h}{2} < \mathrm{Re}\, z \leq h,\ \omega - h \leq \mathrm{Im}\, z < \omega + h \right\}, \tag{6.1}$$

and let

$$\mathcal{T} = \{(2^m, 2^m(1 + 2n)) \mid m, n \in \mathbb{Z}\}.$$

The rectangles $\{Q(h, \omega) \mid (h, \omega) \in \mathcal{T}\}$ form a tiling of $\mathbb{C}_0$, as indicated in Figure 2. Define a partial order $\prec$ on $\mathcal{T}$ by

$$(g, \sigma) \prec (h, \omega) \Longleftrightarrow R(g, \sigma) \subset R(h, \omega).$$

For any bounded set $S \subset \mathbb{C}_0$ we define the infinite matrix $\mu(S)$ by

$$\mu(S) = \sum_{-\lambda_k \in S} b_k^* b_k.$$

Proposition 6.1. *Suppose* $\mathbb{T}$ *is given by* (1.2), (1.3) *and* B *is an infinite matrix with rows* $b_k \in l^{2*}$. *Then* $B \in \mathrm{OCM}(\mathbb{T})$ *if and only if there exists* $M_0 \geq 0$ *such that for any* $(h, \omega) \in \mathcal{T}$,

$$\left\| \sum_{(g,\sigma) \prec (h,\omega)} \mu(Q(g, \sigma)) \right\|_{\mathcal{L}(l^2)} \leq M_0 h\,. \tag{6.2}$$

Proof. The "only if" part is obvious, since

$$\sum_{(g,\sigma)\prec(h,\omega)} \mu(Q(g,\sigma)) = \mu(R(h,\omega)) = \sum_{-\lambda_k \in R(h,\omega)} b_k^* b_k\,,$$

so that (6.2) follows from (1.5).

Conversely, assume (6.2) holds for any $(h,\omega) \in \mathcal{T}$. We wish to show that (1.5) holds for any $h + i\omega \in \mathbb{C}_0$. Thus let $h + i\omega \in \mathbb{C}_0$ and define

$$\mathcal{T}(h,\omega) = \{(g,\sigma) \in \mathcal{T} \mid Q(g,\sigma) \cap R(h,\omega) \neq \emptyset\}\,,$$

and let $\mathcal{T}_m(h,\omega)$ denote the set of maximal elements of $\mathcal{T}(h,\omega)$ (with respect to $\prec$). There can be at most two distinct pairs, say (g_1,σ_1), (g_2,σ_2) in $\mathcal{T}_m(h,\omega)$, since the existence of a third pair would contradict maximality. (If $\mathcal{T}_m(h,\omega)$ contains two pairs (g_1,σ_1) and (g_2,σ_2), then $Q(g_1,\sigma_1)$ and $Q(g_2,\sigma_2)$ are adjacent rectangles of the same size.) Furthermore it is clear that $g_1 = g_2 \leq 2h$. We have

$$R(h,\omega) \subset \bigcup_{(g,\sigma)\in\mathcal{T}(h,\omega)} Q(g,\sigma) = \bigcup_{(g,\sigma)\in\mathcal{T}_m(h,\omega)} R(g,\sigma)\,,$$

whence

$$\begin{aligned}\|\mu(R(h,\omega))\|_{\mathcal{L}(l^2)} &\leq \sum_{(g,\sigma)\in\mathcal{Z}_m(h,\omega)} \|\mu(R(g,\sigma))\|_{\mathcal{L}(l^2)} \\ &\leq 2M_0 h + 2M_0 h\end{aligned}$$

and consequently (1.5) holds with $M = 4M_0$. □

By Theorem 2.5, the estimate (6.2) is necessary for the infinite-time admissibility of B. We do not know if it is sufficient (see (2.10)), however by strengthening (6.2) and adding another assumption we obtain the following sufficient conditions.

Proposition 6.2. *Let* $\mathbb{T}$*,* B *and* μ *be as in Proposition* 6.1*. Assume the following:*
(1) *There exists an* $M_1 \geq 0$ *such that for any* $(h,\omega) \in \mathcal{T}$*,*

$$\sum_{(g,\sigma)\prec(h,\omega)} \|\mu(Q(g,\sigma))\|_{\mathcal{L}(l^2)} \leq M_1 h\,. \tag{6.3}$$

(2) *There exists an* $N \in \mathbb{N}$ *such that each rectangle* $Q(h,\omega)$*, with* $(h,\omega) \in \mathcal{T}$*, contains at most* N *distinct numbers from the set* $\{-\lambda_k \mid k \in \mathbb{N}\}$ *(with arbitrary multiplicity).*

Then $B \in \tilde{\mathcal{B}}(l^2, l^2, \mathbb{T})$*.*

Proof. Let $X = X^* = l^2$ denote the state space and $U = U^* = l^2$ denote the input space. We wish to show that $B \in \tilde{\mathcal{B}}(U, X, \mathbb{T})$. We may assume, without loss of generality, that $N = 1$ (otherwise, X may be decomposed into N $\mathbb{T}$-invariant subspaces and the argument can be applied to each of these separately).

Let $\tau_1, \tau_2, \tau_3 \ldots$ denote an ordering of $\{-\lambda_k \mid k \in \mathbb{N}\}$. Thus, each τ_n equals at least one term of the sequence $(-\lambda_k)$, but it may be equal to infinitely many. The case of only finitely many such τ_n is covered under Theorems 1.2 and 1.3. Indeed, in that case $\mathbb{T}$ is exponentially stable and has a bounded generator (in particular, it is analytic and invertible). Thus, we may assume that (τ_n) is an infinite sequence. Let $(e_k)_{k\in\mathbb{N}}$ be the standard basis for l^2 and for each $n \in \mathbb{N}$ denote

$$X^n = \text{ closed span } \{e_k \mid -\lambda_k = \tau_n\}.$$

When X^n is infinite dimensional, it is isomorphic to l^2. X may be written as the orthogonal direct sum of the spaces X^n:

$$X = X^1 \oplus X^2 \oplus X^3 \oplus \ldots .$$

For each $n \in \mathbb{N}$ let B_n denote the infinite matrix whose kth row is b_k if $-\lambda_k = \tau_n$ and otherwise is the 0 row vector. By hypothesis, the infinite matrix B_n defines a bounded operator from U to X (with range in X^n). Likewise, the conjugate-transpose B_n^* of B_n defines a bounded operator from X to U (with $B_n^* X^j = 0$ for all $j \neq n$). To avoid trivialities we assume that for all $n \in \mathbb{N}$, $B_n \neq 0$.

Let P denote the infinite matrix with entries (p_{jk}) defined by

$$p_{jk} = -\frac{\langle b_j, b_k\rangle}{\lambda_j + \overline{\lambda}_k}.$$

Our goal is to show that P defines a bounded operator on l^2 and conclude from Proposition 5.1 that $B \in \tilde{\mathcal{B}}(l^2, l^2, \mathbb{T})$. P may be rewritten as

$$P = \sum_{m,n=1}^{\infty} \frac{B_m B_n^*}{\tau_m + \overline{\tau}_n}.$$

Let I denote the identity operator on U and let $\mathbf{K} = P^0 \otimes I$ (the infinite Kronecker product), where $P^0 = (p^0_{mn})$ is defined by

$$p^0_{mn} = \frac{\|B_m\| \cdot \|B_n\|}{\tau_m + \overline{\tau}_n}.$$

Thus $\mathbf{K}$ can be viewed in a natural way as a partitioned infinite matrix in which each block is a multiple of the infinite identity matrix. P may be factored as follows:

$$P = \left(\frac{B_1}{\|B_1\|}, \frac{B_2}{\|B_2\|}, \ldots\right) \mathbf{K} \left(\frac{B_1}{\|B_1\|}, \frac{B_2}{\|B_2\|}, \ldots\right)^* .$$

Denoting $\mathbf{B} = \left(\frac{B_1}{\|B_1\|}, \frac{B_2}{\|B_2\|}, \ldots\right)$, we can write this more compactly: $P = \mathbf{B}\mathbf{K}\mathbf{B}^*$. The matrix $\mathbf{B}$ is partitioned into infinite blocks arranged in an infinite row and multiplication in the last identity is defined blockwise.

Let $l^2(l^2)$ denote the Hilbert space of l^2-valued sequences with the l^2 norm (so that $l^2(l^2)$ is isomorphic to l^2). Then $\mathbf{K}$ may be regarded as an operator on $l^2(l^2)$ (possibly unbounded). For each $n \in \mathbb{N}$, the operator $B_n/\|B_n\|$ has norm 1 as an operator from U to X and has range in X^n. It follows from the orthogonality of these subspaces that $\mathbf{B}$ has norm 1 as an operator from $l^2(l^2)$ to X. Likewise, $\mathbf{B}^*$ has norm 1 as an operator from X to $l^2(l^2)$. Thus we obtain that $\|P\| \leq \|\mathbf{K}\|$. It remains to show that $\mathbf{K}$ is a bounded operator on $l^2(l^2)$, i.e., $\|\mathbf{K}\| < \infty$.

Since P^0 and $\mathbf{K}$ are Hermitian, their norms are equal to their spectral radii. Since the spectral radius of the Kronecker product of two Hermitian matrices is the product of the spectral radii of each matrix (see, e.g., Lancaster and Tismenetsky [10, pp. 411, 412], easily extended to infinite matrices), it follows that $\|\mathbf{K}\| = \|P^0\|$, so that $\|P\| \leq \|P^0\|$. Thus, we have to show that $\|P^0\| < \infty$.

To prove the boundedness of P^0, let $\mathbf{S} = (\mathbf{S}_t)_{t\geq 0}$ be the diagonal semigroup with n-th diagonal element defined by $(\mathbf{S}_t)_n = e^{-\tau_n t}$, and let β denote the sequence $(\|B_n\|)_{n\in\mathbb{N}}$. Let $(h,\omega) \in \mathcal{T}$ be such that $Q(h,\omega)$ contains a term τ_n from the sequence $\tau_1, \tau_2, \tau_3, \ldots$. (This rectangle can not contain more than one term of the sequence, by our assumption that $N = 1$.) Since $\mu(Q(h,\omega)) = B_n^* B_n$, we have

$$\|\mu(Q(h,\omega))\|_{\mathcal{L}(l^2)} \;=\; \|B_n\|^2 \;=\; \beta_n^2\,.$$

Now, the inequality (6.3) implies that β satisfies the Carleson measure criterion (2.8) for $\mathbf{S}$. (Actually, we only know that (2.8) is satisfied for pairs (h,ω) which are in $\mathcal{T}$, but by Proposition 6.1 this is enough.) Thus, by Theorem 2.3, β is infinite-time admissible for $\mathbf{S}$. Now Proposition 5.1 shows that P^0 is bounded on l^2, since it is the controllability Gramian of $\mathbf{S}$ and β. □

Definition 6.3. A set $S \subset \mathbb{C}_0$ is called *σ-rectangular* if there is an at most countable set $\mathcal{I} \subset (0,\infty) \times \mathbb{R}$ such that (with the notation in (6.1))

$$S = \bigcup_{(h,\omega)\in\mathcal{I}} Q(h,\omega)\,, \tag{6.4}$$

and for any distinct $(g,\sigma), (h,\omega) \in \mathcal{I}$

$$[\sigma - g, \sigma + g) \cap [\omega - h, \omega + h) = \emptyset\,.$$

Figure 3 shows an intuitive picture of a σ-rectangular set.

Proposition 6.4. *Let $\mathbb{T}$ be as in (1.3) and such that $\{-\lambda_k \mid k \in \mathbb{N}\} \subset S$, where S is σ-rectangular, as in (6.4). Then $b \in \tilde{\mathfrak{b}}(l^2, \mathbb{T})$ if and only if there exists $M \geq 0$ such that for every $(h,\omega) \in \mathcal{I}$,*

$$\sum_{-\lambda_k \in Q(h,\omega)} |b_k|^2 \;\leq\; Mh\,. \tag{6.5}$$

Proof. The necessity of (6.5) is clear from Theorem 2.3. To prove sufficiency, for any $h + i\omega \in \mathbb{C}_0$ denote

$$\mathcal{I}(h,\omega) = \{(g,\sigma) \in \mathcal{I} \mid Q(g,\sigma) \cap R(h,\omega) \neq \emptyset\}\,.$$

Fig. 2. The tiling $\mathcal{T}$ of $\mathbb{C}_0$. Fig. 3. A σ-rectangular set.

Then from simple geometric considerations

$$\sum_{(g,\sigma)\in\mathcal{I}(h,\omega)} g \leq 5h\,.$$

Thus,

$$\sum_{-\lambda_k\in R(h,\omega)} |b_k|^2 \leq \sum_{(g,\sigma)\in\mathcal{I}(h,\omega)} \sum_{-\lambda_k\in Q(g,\sigma)} |b_k|^2$$
$$\leq \sum_{(g,\sigma)\in\mathcal{I}(h,\omega)} Mg \leq 5Mh\,,$$

so that by Theorem 2.3, b is infinite-time admissible for $\mathbb{T}$. □

Remark 6.5. With $\mathbb{T}$ and S as in Proposition 6.4 and B an infinite matrix with rows in l^{2*}, to check that $B\in$ OCM ($\mathbb{T}$) it is enough to verify (1.5) for the pairs $(h,\omega)\in\mathcal{I}$ used in the definition (6.4) of S. The proof is similar to that of Proposition 6.4.

Proposition 6.6. *Let* $\mathbb{T}$ *be as in* (1.3) *and assume the following:*
(1) *We have* $\{-\lambda_k \mid k\in\mathbb{N}\}\subset S$*, where* S *is a* σ*-rectangular set, as in* (6.4)*.*
(2) *There exists an* $N\in\mathbb{N}$ *such that each rectangle* $Q(h,\omega)$*, with* $(h,\omega)\in\mathcal{I}$*, contains at most* N *distinct numbers from the set* $\{-\lambda_k \mid k\in\mathbb{N}\}$ *(with arbitrary multiplicity).*

Then $\tilde{\mathcal{B}}(l^2,l^2,\mathbb{T}) = \mathrm{OCM}(\mathbb{T})$*.*

Proof. We have $\tilde{\mathcal{B}}(l^2,l^2,\mathbb{T})\subset \mathrm{OCM}(\mathbb{T})$, by Theorem 2.5. Conversely, suppose $B\in \mathrm{OCM}(\mathbb{T})$ and let $\tau_1,\tau_2,\tau_3...$ denote an ordering of $\{-\lambda_k \mid k\in\mathbb{N}\}$. Now we argue as in the proof of Proposition 6.2: we may assume that (τ_n) is an infinite sequence, we introduce the infinite matrices B_n, P, P^0, $\mathbf{K}$, $\mathbf{B}$ (such that $P=\mathbf{BKB}^*$ and $\mathbf{K}=P^0\otimes I$), the semigroup $\mathbf{S}$ (such that $(\mathbf{S}_t)_n = e^{-\tau_n t}$) and the sequence $\beta = (\|B_n\|)_{n\in\mathbb{N}}$. Again we have $\|P\|\leq\|\mathbf{K}\| = \|P^0\|$, so that B is infinite-time admissible for $\mathbb{T}$ if β is infinite-time admissible for $\mathbf{S}$. It remains to prove this fact about β. For any $(h, \omega) \in \mathcal{I}$ for which $Q(h, \omega)$ contains a term τ_n, we have by (1.5)

$$\beta_n^2 = \|B_n^* B_n\| = \Big\| \sum_{-\lambda_k \in Q(h,\omega)} b_k^* b_k \Big\|_{\mathcal{L}(l^2)} \leq Mh .$$

It follows by Proposition 6.4 that β is infinite-time admissible for $\mathbf{S}$. $\square$

Both Propositions 6.2 and 6.6 contain a similar condition (2), which prevents the eigenvalues from being too dispersed. This condition can be relaxed by perturbing the eigenvalues, using Propositions 4.3, 4.4 and 4.7. Unfortunately, we do not know if it is possible to get rid of condition (2) in general. In the particular case of the eigenvalues lying in $-S$, where S is as in (4.18), this is possible, see Theorem 6.8.

The following result is a generalization of the part of Theorem 1.3 relating to analytic semigroups since there, the semigroup was assumed to be *exponentially* stable.

Proposition 6.7. *Let* $\mathbb{T}$ *be as in* (1.3) *and assume that for some* $\beta \in \mathbb{R}$ *and some* $\theta \in (0, \pi/2)$, $\{-\lambda_k \mid k \in \mathbb{N}\} \subset S$, *where* $S = \{s \in \mathbb{C}_0 \mid |\arg(s - i\beta)| \leq \theta\}$.

Then $\tilde{\mathcal{B}}(l^2, l^2, \mathbb{T}) = \mathrm{OCM}(\mathbb{T})$.

Proof. We have $\tilde{\mathcal{B}}(l^2, l^2, \mathbb{T}) \subset \mathrm{OCM}(\mathbb{T})$, by Theorem 2.5. Conversely, suppose $B \in \mathrm{OCM}(\mathbb{T})$. We denote by $(e_k)_{k\in\mathbb{N}}$ the standard basis of l^2. For each $n \in \mathbb{Z}$ define $\Lambda^n = \{-\lambda_k \,|\, 2^n \leq -\mathrm{Re}\,\lambda_k < 2^{n+1}\}$ and $X^n =$ closed span$\{e_k \mid -\lambda_k \in \Lambda^n\}$. Let P_n be the orthogonal projection of l^2 onto the $\mathbb{T}$-invariant subspace X^n, so that $\|x\|^2 = \sum_{n\in\mathbb{Z}} \|P_n x\|^2$ holds for all $x \in l^2$.

Now we can argue as in the proof of Theorem 1.3 in [8] to show that there exists a $K \geq 0$ such that for every $u \in L^2([0, \infty), l^2)$

$$\begin{aligned} \|\tilde{\Phi} u\|^2 &= \sum_{n\in\mathbb{Z}} \left\| \int_0^\infty \mathbb{T}_t P_n B u(t)\, dt \right\|^2 \\ &\leq K \sum_{n\in\mathbb{Z}} \left(\int_0^\infty e^{-2^n} 2^{n/2} \|u(t)\| dt \right)^2 . \end{aligned}$$

The only difference with respect to the argument in [8] is that the index set $\{0, 1, 2, ...\}$ has been replaced by $\mathbb{Z}$. Lemma 3.8 from [8] can be reformulated (using Theorem 2.3) such that it will work for the index set $\mathbb{Z}$. Combining our last estimate with this (reformulated) lemma yields that there exists $k \geq 0$ such that

$$\|\tilde{\Phi} u\| \leq k \|u\|_{L^2} ,$$

i.e., B is infinite-time admissible for $\mathbb{T}$. $\square$

Now we reformulate and prove part (iii) of Theorem 1.5.

Theorem 6.8. *Let* $\mathbb{T}$ *be as in* (1.3) *and such that* $\{-\lambda_k \mid k \in \mathbb{N}\} \subset S$, *where* S *is as in* (4.18), *with* $\alpha \geq 0$ *(i.e.,* $|\mathrm{Re}\,\lambda_k| \sim |\mathrm{Im}\,\lambda_k|^\alpha$*). Then* $\tilde{\mathcal{B}}(l^2, l^2, \mathbb{T}) = \mathrm{OCM}(\mathbb{T})$.

Proof. We have $\tilde{\mathcal{B}}(l^2, l^2, \mathbb{T}) \subset \mathrm{OCM}(\mathbb{T})$, by Theorem 2.5. Conversely, suppose $\alpha < 1$ and $B \in \mathrm{OCM}(\mathbb{T})$. We partition S into two sets S_0 and S_1, according to $|\mathrm{Im}\,z| <$

1 and $|\mathrm{Im}\, z| \geq 1$, respectively. (With the notation from the proof of Theorem 4.7, $S_1 = S_- \cup S_+$.) Now l^2 can be decomposed into the orthogonal sum of the $\mathbb{T}$-invariant subspaces X^0 and X^1, where X^0 corresponds to $-\lambda_k \in S_0$ and similarly for X^1.

Denoting by $\mathbb{T}^0$ and $\mathbb{T}^1$ the restrictions of $\mathbb{T}$ to X^0 and to X^1, we can partition $\mathbb{T}$ and B as follows:

$$\mathbb{T}_t = \begin{bmatrix} \mathbb{T}_t^0 & 0 \\ 0 & \mathbb{T}_t^1 \end{bmatrix}, \qquad B = \begin{bmatrix} B^0 \\ B^1 \end{bmatrix}.$$

It will be sufficient to prove that $B^0 \in \tilde{\mathcal{B}}(l^2, X^0, \mathbb{T}^0)$ and $B^1 \in \tilde{\mathcal{B}}(l^2, X^1, \mathbb{T}^1)$. We start with B^0, which is easier. If X^0 is finite dimensional, then $\mathbb{T}^0$ is exponentially stable and there is nothing to prove. If X^0 is infinite dimensional, then it is isomorphic to l^2, $\mathbb{T}^0$ is diagonal and $B^0 \in \mathrm{OCM}(\mathbb{T}^0)$. There exists a $\theta \in (0, \pi/2)$ such that $|\arg z| \leq \theta$ for all $z \in S_0$. By Proposition 6.7, B^0 is infinite-time admissible for $\mathbb{T}^0$.

Now let us prove the infinite-time admissibility of B^1. Again, we may assume without loss of generality that $X^1 = l^2$, $\mathbb{T}^1$ is diagonal and $B^1 \in \mathrm{OCM}(\mathbb{T}^1)$. By a geometric argument which we leave to the reader, it can be shown that S_1 is contained in a finite union of σ-rectangular sets (their number increases as b/a increases). Without loss of generality we may assume that it is just one σ-rectangular set (this would be true if b/a were close to 1):

$$S_1 \subset \bigcup_{(h,\omega)\in\mathcal{I}} Q(h,\omega), \tag{6.6}$$

with the intervals $[h-\omega, h+\omega)$ mutually disjoint, as in (6.4).

We denote the eigenvalues of A^1, the generator of $\mathbb{T}^1$, by $\mu_1, \mu_2, \ldots$. By another geometric argument left to the reader, it can be shown that we can find 13 points in each rectangle $Q(h,\omega)$ such that the distance from any other point in $Q(h,\omega)$ to the closest of these 13 points is $\leq h/4$ (we do not claim that 13 is minimal in this respect). This makes it possible to construct a sequence $\nu_1, \nu_2, \ldots$ in $\mathbb{C}$ with the following properties:

(1) $-\nu_k \in S_1$,

(2) $-\nu_k$ and $-\mu_k$ are in the same rectangle $Q(h,\omega)$ from (6.6),

(3) $|\nu_k - \mu_k| \leq \frac{h}{4}$, with h as in (2) above,

(4) there are at most 13 distinct numbers $-\nu_k$ in each rectangle $Q(h,\omega)$ from (6.6).

Indeed, we can construct first a sequence $\xi_1, \xi_2, \ldots$ satisfying (2), (3) and (4) and then define $-\nu_k$ as the closest point to $-\xi_k$ in the set $Q(h,\omega) \cap S_1$.

Let $A^2 = \mathrm{diag}\,(\nu_1, \nu_2, \ldots)$, with its natural domain, and let $\mathbb{T}^2$ denote the diagonal semigroup generated by A^2. Since $B^1 \in \mathrm{OCM}(\mathbb{T}^1)$, by Remark 6.5 it follows that $B^1 \in \mathrm{OCM}(\mathbb{T}^2)$. Now by Proposition 6.6 (with $N = 13$) it follows that

$$B^1 \in \tilde{\mathcal{B}}(l^2, l^2, \mathbb{T}^2).$$

We denote

$$\beta_k = \left(\frac{1}{2}\,|\mathrm{Re}\,\nu_k|\right)^{1/2}, \qquad B^2 = \mathrm{diag}\,(\beta_1, \beta_2, \ldots),$$

as in (4.9) and (4.10), with ν_k in place of λ_k. Thus, $B^2 \in \mathrm{OCM}(\mathbb{T}^2)$ and, by Theorem 2.6 (with $m = 1$), $B^2 \in \tilde{\mathcal{B}}(l^2, l^2, \mathbb{T}^2)$.

Let the sequence (c_k) in $\mathbb{C}$ be such that $\mu_k = \nu_k + \beta_k c_k$. For some $k \in \mathbb{N}$, let $Q(h,\omega)$ be the rectangle from (6.6) containing $-\nu_k$. Then

$$|c_k|^2 = \left|\frac{\mu_k - \nu_k}{\beta_k}\right|^2 \leq \frac{(h/4)^2}{|\mathrm{Re}\,\nu_k|/2} \leq \frac{(h/4)^2}{h/4} = \frac{h}{4} \leq \frac{1}{2}|\mathrm{Re}\,\nu_k|\,.$$

Thus, the sequence (c_k) satisfies (4.10) (with ν_k in place of λ_k). We define the operator $C = \mathrm{diag}\,(c_1, c_2, ...)$, as in (4.9). Since $C^* \in \mathrm{OCM}\,(\mathbb{T}^{2*})$, by Theorem 2.6 C is an admissible observation operator for $\mathbb{T}^2$.

By Theorem 4.7 (with A^2 and B^1 in place of A and B_1) we have that (A^2, B^1, C) is a regular triple. Thus, by Proposition 4.3, B^1 is admissible for the semigroup $\mathbb{T}^+$ generated by

$$A^+ = \mathrm{diag}\,(\nu_1 + \beta_1 c_1, \nu_2 + \beta_2 c_2, ...)\,.$$

But since $\nu_k + \beta_k c_k = \mu_k$, we see that $A^+ = A^1$ and $\mathbb{T}^+ = \mathbb{T}^1$, so that $B^1 \in \mathcal{B}(l^2, l^2, \mathbb{T}^1)$. Finally, since $\mathbb{T}^1$ is exponentially stable, we have that $\mathcal{B}(l^2, l^2, \mathbb{T}^1) = \tilde{\mathcal{B}}(l^2, l^2, \mathbb{T}^1)$. Thus we have finished the proof for $\alpha < 1$.

The case $\alpha = 1$ is a particular case of Proposition 6.7.

The case $\alpha > 1$ can be reduced to the case $\alpha < 1$, using the following fact: B is infinite-time admissible for the semigroup generated by A, if and only if $A^{-1}B$ is infinite-time admissible for the semigroup generated by A^{-1}. This follows from Theorem 3.1, by multiplying equation (3.2) with A^{-1} to the left and with A^{-*} to the right. □

From Propositions 6.6, 6.7 and 6.8 we know that for semigroups given by (1.2), (1.3) we have $\tilde{\mathcal{B}}(l^2, l^2, \mathbb{T}) = \mathrm{OCM}\,(\mathbb{T})$ in several situations not covered under Theorems 1.2 and 1.3. These results can be summarized as follows:

Proposition 6.9. *Let $\mathbb{T}$ be as in (1.3), with $\{-\lambda_k \mid k \in \mathbb{N}\} \subset S \subset \mathbb{C}_0$.*

Then $\tilde{\mathcal{B}}(l^2, l^2, \mathbb{T}) = \mathrm{OCM}\,(\mathbb{T})$ if one of the following conditions holds:

(C1) *$S = \{z \in \mathbb{C}_0 \mid a < \mathrm{Re}\, z < b\}$, with $0 < a < b$;*
(C2) *$S = \{z \in \mathbb{C}_0 \mid |\arg(z - i\beta)| \leq \psi\}$, with $\psi < \frac{\pi}{2}$ and $\beta \in \mathbb{R}$;*
(C3) *$S = \{z \in \mathbb{C}_0 \mid a|\mathrm{Im}\, z|^\alpha \leq \mathrm{Re}\, z \leq b|\mathrm{Im}\, z|^\alpha\}$, with $\alpha \geq 0$ and $0 < a < b$;*
(C4) *S is a σ-rectangular set, and condition (2) from Proposition 6.6 holds;*
(C5) *S is a finite union of sets which satisfy (C1), (C2), (C3) or (C4).*

(C1) is a particular case of (C3), but we found it more clear to list it separately. Combinations of (C1) and (C2) occur for example in some thermoelastic systems (see Hansen [7] and Leis [11, Chapter 13]). Spectral arrangements which are as in (C3) can occur in certain models for internally damped beams and plates (see S. Chen and Triggiani [3], and G. Chen and Russell [2]). We shall discuss such an example in a separate article.

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