A DYNAMICAL MODEL FOR MULTILAYERED PLATES WITH INDEPENDENT SHEAR DEFORMATIONS

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Abstract. In this paper a dynamic model for an n-layered plate is developed based upon the assumptions of Reissner-Mindlin plate theory. Each plate layer is assumed to be transversely isotropic, transversely homogeneous and of a uniform thickness, however, no symmetry in the material properties or thicknesses of each plate is assumed. The layers are assumed to be perfectly bonded so that no slip occurs along the interface. No additional *a-priori* kinematic restrictions are imposed upon the motion of the plates. The equations of motion are derived by the principle of virtual work. Existence and uniqueness results are obtained. In the case where the layers are symmetric we show that all solutions decouple into a bending solution (with antisymmetric displacements about the mid-plane) and an in-plane solution (with symmetric displacements).

1. Introduction. The first model for a thin, multilayered plate which includes the effects of transverse shear deformation is probably due to Reissner [11], where a static model is developed for a sandwich plate consisting of an inner core bonded to two symmetric face plates. Since then, numerous dynamic models for multilayered materials based upon similar ideas have been proposed (see e.g., [1], [3], [4], [8], [14], [15] and the references therein). In many of these models, (often referred to as *constrained layer models*) the inner core is assumed to be much more flexible than the face material so that bending stresses can be ignored within the core and the transverse shear deformations can be assumed negligible within the face plates. Thus in the usual approach to modeling sandwich plates, the outer layers are modeled by plates which allow little or no shear while the inner layer is modeled as a material in which only shear stresses are considered relevant. From an applications viewpoint, a goal has been to identify *effective* elastic modulii for the composite plate so that a low-order system involving only the transverse displacement (or

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possibly transverse displacement and effective rotations) can be obtained. In order to accomplish this, the number of degrees of freedom of a plate element are artificially reduced by relating the the stresses and/or strains in each layer with one another.

In this paper we develop a model for the motion of a multilayered plate which allows independent shear rotations in each layer. Each plate layer is assumed to be transversely isotropic, homogeneous in the transverse direction and of a uniform thickness, however, the material properties and thicknesses of each layer can be different. Each layer is modeled under the assumptions of Reissner-Mindlin plate theory [9,10,11], i.e., within each layer the in-plane displacements vary linearly with respect to the transverse coordinate and the transverse normal stress vanishes throughout the thickness. The surfaces of each layer are perfectly bonded so that no slip occurs along the interfaces, however, a main point is that we do not impose additional *a-priori* kinematic restrictions which couple stresses and/or strains of each layer with one another.

The equations of motion are derived through the principle of virtual work and we discuss the modifications needed to include strain-rate damping. The resulting system is shown to be well-posed in an appropriate function space.

In the case where the layers are symmetric about the center sheet of the plate (that is, the thicknesses, densities and elastic parameters are symmetric with respect to the middle layer or middle interface, as the case may be) we show that the motions decouple in a *bending solution*, with displacements which are antisymmetric about the center sheet and a *stretching solution*, with displacements that are symmetric about the center sheet. This type of decoupling has been observed in three-dimensional plate theories as well [13].

A similar model is obtained in [3], where the equations for a symmetric three-layer beam are derived under the assumption that the motion is antisymmetric about the centerline of the beam and Poisson effects are negligible. A fourth order beam model then is obtained when continuity of shear stresses across the interfaces is imposed and rotational inertia is ignored. As mentioned, in the symmetric case we obtain solutions with antisymmetric displacements about the center sheet, however this is due to the decoupling and is not imposed *a-priori*. The assumption of shear stress continuity along the interfaces leads to a model in which these stresses are constant throughout the thickness and is not assumed here.

2. Basic assumptions. Our plate consists of n plate layers which occupy the region $\Omega \times (0,h)$ at equilibrium, where Ω is an open bounded domain in $\mathbb{R}^2$ with sufficiently smooth (say C^1) boundary Γ. Let

$$0 = z_0 < z_1 < \ldots < z_{n-1} < z_n = h, \qquad h_i = z_i - z_{i-1},\ i = 1,2,\ldots,n.$$

We use the rectangular coordinates $\underline{x} = \{x_1, x_2\}$ to denote points in Ω and $x = \{\underline{x}, x_3\} = \{x_1, x_2, x_3\}$ to denote points in $Q = \cup_{i=1}^{n} Q_i$, where

$$Q_i = \Omega \times (z_{i-1}, z_i), \qquad i = 1,2,\ldots,n.$$

For $x \in Q$ let $U(x) = \{U_1, U_2, U_3\}(x)$ denote the displacement vector of the point which, when the plate in equilibrium has coordinates $x = \{x_1, x_2, x_3\}$. (We suppress all time dependence where there is no possiblility of confusion.) In addition let us define $u^i = \{u_1^i, u_2^i\}$ and u_3^i, $i = 0,1,2,\ldots n$ by

$$u_j^i(\underline{x}) = U_j(\underline{x}, z_i) \quad j = 1,2,3, \quad \forall \underline{x} \in \Omega.$$

Throughout this paper we will continue to reserve the index i to refer to the particular layer or interface within the composite plate. For *vector* quantities whose components vary from layer to layer, the index i will be *superscripted*, while for *scalar* quantities the i will be *subscripted*.

2.1. Stress-strain relations. Let $\sigma_{jk}, \epsilon_{jk}$ $(j,k = 1,2,3)$ denote the stress and strain tensors, respectively. For a small displacement theory we assume

$$\epsilon_{jk}(x) = \frac{1}{2}\left(\frac{\partial U_j(x)}{\partial x_k} + \frac{\partial U_k(x)}{\partial x_j}\right), \quad \forall\, x \in Q. \tag{2.1}$$

For a homogeneous, transversely isotropic material we have within each layer (see Reissner, [12])

$$\begin{cases} \epsilon_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} - \frac{\nu_z}{\sqrt{EE_z}}\sigma_{33} & \epsilon_{12} = \frac{1+\nu}{E}\sigma_{12} \\ \epsilon_{22} = -\frac{\nu}{E}\sigma_{11} + \frac{1}{E}\sigma_{22} - \frac{\nu_z}{\sqrt{EE_z}}\sigma_{33} & \epsilon_{13} = \frac{1}{2G_z}\sigma_{13} \\ \epsilon_{33} = -\frac{\nu_z}{\sqrt{EE_z}}\sigma_{11} - \frac{\nu_z}{\sqrt{EE_z}}\sigma_{22} + \frac{1}{E}\sigma_{33} & \epsilon_{23} = \frac{1}{2G_z}\sigma_{23}, \end{cases} \tag{2.2}$$

where E, E_z denote the Young's moduli (the z subscript refers to the transverse direction), ν, ν_z denote the Poisson's ratios, $E/2(1+\nu)$ denotes the in-plane shear modulus and G_z denotes the transverse shear modulus. All the elastic moduli are allowed to depend upon the coordinate $\underline{x}$ and the layer i (but otherwise are independent of x_3) and are assumed to be bounded above and below by positive numbers. The Poisson's ratios are less than 1/2. The isotropic case is obtained from (2.2) by setting $E_z = E$, $\nu_z = \nu$ and $G_z = E/2(1+\nu)$.

Following Mindlin's approach [9], σ_{33} is assumed to be negligible so that ϵ_{33} may be expressed in terms of the other principle strains:

$$\epsilon_{33} = \frac{-\nu_z\sqrt{E}}{(1-\nu)\sqrt{E_z}}(\epsilon_{11}+\epsilon_{22}). \tag{2.3}$$

From (2.2) and (2.3) we have

$$\begin{cases} \sigma_{11} = \frac{E}{1-\nu^2}(\epsilon_{11}+\nu\epsilon_{22}) & \sigma_{12} = \frac{E}{1+\nu}\epsilon_{12} \\ \sigma_{22} = \frac{E}{1-\nu^2}(\nu\epsilon_{11}+\epsilon_{22}) & \sigma_{13} = 2G_z\epsilon_{13} \\ \sigma_{33} = 0 & \sigma_{23} = 2G_z\epsilon_{23}. \end{cases} \tag{2.4}$$

2.2. Displacement assumption. To obtain a two-dimensional plate model, in addition to (2.4), some assumptions are needed on either the form of the displacements or the form of the stresses as a function of the transverse coordinate. Reissner's approach [10,11] is based on first order approximations for the stresses while Mindlin's approach [9] is based on linear displacement assumptions. For example, Reissner assumes σ_{13} and σ_{23} to be parabolic, in such a way that these stresses vanish on the surfaces of the plate. In Mindlin's approach ϵ_{13} and ϵ_{23} are assumed to be constant throughout the thickness (consequently the stresses will not in general vanish on the surfaces), however, a *shear correction coefficient* κ is incorporated in (2.4) to compensate:

$$\sigma_{13} = 2G\epsilon_{13}, \quad \sigma_{23} = 2G\epsilon_{23}; \qquad G = \kappa G_z. \tag{2.5}$$

When $\kappa = 5/6$, to highest order in the thickness, both approaches lead to the same two-dimensional plate theory, although the displacement variables have slightly different meanings with respect to the transverse direction (see Reissner's survey article [12] and also [9] for comparisons of these plate theories).

In our approach to modeling the multilayer plate, we assume (2.4) with the correction (2.5) in conjunction with linear displacement assumptions in each layer. In addition, the no-slip condition imposes continuity of the displacements along the interfaces so that (usually) corners exist at the interfaces. Thus by specifying the displacements $u^i(\underline{x})$, $u_3^i(\underline{x})$, $i = 0,1,2,\ldots,n$ at each $\underline{x}$ in Ω, the displacement U is uniquely determined. The assumption that the transverse normal stresses vanish throughout the thickness implies

$$w(\underline{x}) \equiv u_3^0(\underline{x}) = u_3^1(\underline{x}) = \ldots = u_3^n(\underline{x}) \quad \forall\, \underline{x} \in \Omega.$$

Thus we find that each filament that is originally orthogonal to the surfaces has $2n+3$ degrees of freedom: w, $u^0, \ldots, u^n$.

For $i = 1, 2, \ldots, n$ define $\psi^i = \{\psi^i_1, \psi^i_2\}$, $\varphi^i = \{\varphi^i_1, \varphi^i_2\}$, and $v^i = \{v^i_1, v^1_2\}$ by

$$\psi^i = \frac{u^i - u^{i-1}}{h_i}, \quad \varphi^i = \psi^i + (\nabla w)^T, \quad v^i = \frac{u^{i-1} + u^i}{2}. \tag{2.6}$$

The components ψ^i_j of ψ^i can be viewed as the *total rotation angles* of filament within the i-th layer in the x_j-x_3 plane (with negative orientation). The components of φ represent the (small angle approximation for the) *shear angles* within each layer. (See Figure 1.) The components of v^i represent the in-plane displacements of the midplanes of the i-th layer.

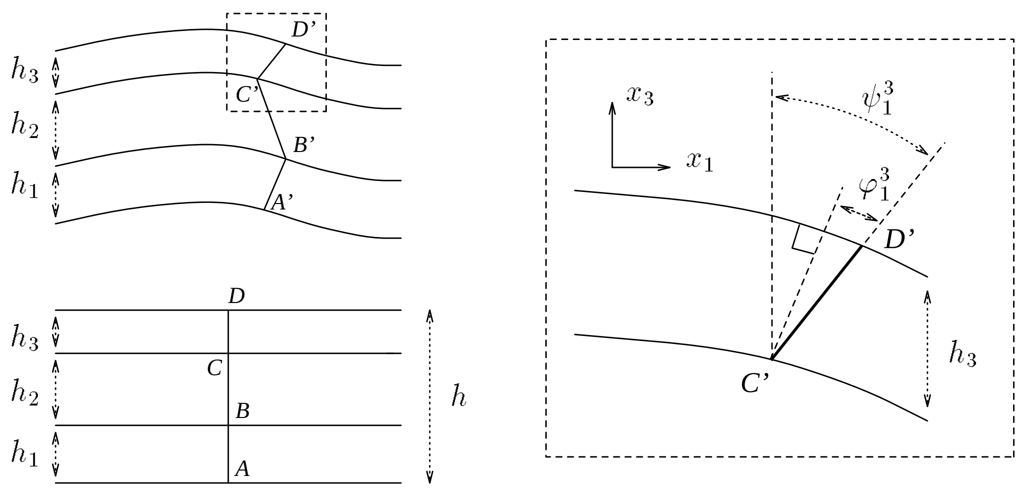

Fig. 1. A projection onto the x_1-x_3 axis of the deformation of the straight filament A-B-C-D at equilibrium (bottom left) into the deformed filament A'-B'-C'-D'. If C and D have coordinates $\{\underline{x}, z_2\}$ and $\{\underline{x}, z_3\}$ then C' and D' have coordinates $\{\underline{x} + u^2(\underline{x}), z_2 + w(\underline{x})\}$ and $\{\underline{x} + u^3(\underline{x}), z_3 + w(\underline{x})\}$, respectively.

Define

$$\hat{z}_i = (z_{i-1} + z_i)/2.$$

The displacements within the ith layer can be written in terms of the translations v^i and total rotation angles ψ^i as

$$\begin{cases} U_1(\underline{x}, x_3) &= v^i_1(\underline{x}) + (x_3 - \hat{z}_i)\psi^i_1(\underline{x}) \quad z_{i-1} < x_3 < z_i \\ U_2(\underline{x}, x_3) &= v^i_2(\underline{x}) + (x_3 - \hat{z}_i)\psi^i_2(\underline{x}) \quad z_{i-1} < x_3 < z_i \\ U_3(\underline{x}, x_3) &= w(\underline{x}) \qquad\qquad\qquad\qquad z_{i-1} < x_3 < z_i. \end{cases} \tag{2.7}$$

Note that continuity of the displacements along the interfaces follows from way that ψ^i and v^i, $i = 1, 2, \ldots, n$ are related in (2.6).

Substituting (2.7) into (2.1) gives an expression for the strain within the i-th layer:

$$\begin{cases} \epsilon_{11} &= \frac{\partial v_1^i}{\partial x_1} + (x_3 - \hat{z}_i)\frac{\partial \psi_1^i}{\partial x_1} \quad \epsilon_{22} = \frac{\partial v_2^i}{\partial x_2} + (x_3 - \hat{z}_i)\frac{\partial \psi_2^i}{\partial x_2} \\ \epsilon_{12} &= \frac{1}{2}\left[\frac{\partial v_1^i}{\partial x_2} + \frac{\partial v_2^i}{\partial x_1} + (x_3 - \hat{z}_i)\left(\frac{\partial \psi_1^i}{\partial x_2} + \frac{\partial \psi_2^i}{\partial x_1}\right)\right] \\ \epsilon_{13} &= \frac{1}{2}(\varphi_1^i) \qquad \epsilon_{23} = \frac{1}{2}(\varphi_2^i). \end{cases} \tag{2.8}$$

Since we have assumed σ_{33} to be negligible we may assume (for the purpose of calculating the energy) that

$$\epsilon_{33} = 0.$$

2.3. Strain and kinetic energy. The strain energy $\mathcal{P} = \sum_{i=1}^n \mathcal{P}_i$ for the composite plate is given by

$$\mathcal{P}_i = \frac{1}{2}\int_{Q_i} \sum_{j,k=1}^{3} \epsilon_{jk}\sigma_{jk}\,d\underline{x}\,dx_3.$$

From (2.4) and (2.5) $\mathcal{P}_i$ can be written in terms of the strains:

$$\mathcal{P}_i = \frac{1}{2}\int_{Q_i} \frac{E}{(1-\nu^2)}\left(\epsilon_{11}^2 + 2\nu\epsilon_{11}\epsilon_{22} + \epsilon_{22}^2 + 2(1-\nu)\epsilon_{12}^2\right) + 4G(\epsilon_{13}^2 + \epsilon_{23}^2)\,d\underline{x}\,dx_3.$$

From (2.8) we obtain

$$\begin{aligned} \mathcal{P}_i = \frac{h_i^3}{2}\int_\Omega D_i &\left[\left(\frac{\partial \psi_1^i}{\partial x_1}\right)^2 + \left(\frac{\partial \psi_2^i}{\partial x_2}\right)^2 + 2\nu_i\left(\frac{\partial \psi_2^i}{\partial x_2}\frac{\partial \psi_1^i}{\partial x_1}\right)\right. \\ &\left.+ \left(\frac{1-\nu_i}{2}\right)\left(\frac{\partial \psi_1^i}{\partial x_2} + \frac{\partial \psi_2^i}{\partial x_1}\right)^2\right] d\underline{x} \\ + \frac{h_i}{2}\int_\Omega 12D_i &\left[\left(\frac{\partial v_1^i}{\partial x_1}\right)^2 + \left(\frac{\partial v_2^i}{\partial x_2}\right)^2 + 2\nu_i\left(\frac{\partial v_1^i}{\partial x_1}\frac{\partial v_2^i}{\partial x_2}\right)\right. \\ &\left.+ \left(\frac{1-\nu_i}{2}\right)\left(\frac{\partial v_1^i}{\partial x_2} + \frac{\partial v_2^i}{\partial x_1}\right)^2\right] + G_i((\varphi_1^i)^2 + (\varphi_2^i)^2)\,d\underline{x} \end{aligned}$$

where $\nu_i = \nu(\cdot, i)$, $G_i = G(\cdot, i)$, and $D_i = E(\cdot, i)/12(1-\nu_i^2)$. $D_i h_i^3$ is the *modulus of flexural rigidity* (see [6, p. 10]) for the ith layer and $h_i G_i$ is the *modulus of elasticity in shear* (see [6, p. 14]) for the ith layer.

The kinetic energy $\mathcal{K} = \sum_{i=1}^n \mathcal{K}_i$ is defined by

$$\mathcal{K}_i = \frac{1}{2}\int_{Q_i} \rho_i(\dot{U}_1^2 + \dot{U}_2^2 + \dot{U}_3^2)\,d\underline{x}\,dx_3,$$

where $\dot{} = d/dt$ and $\rho_i = \rho_i(\underline{x}) > c_\rho > 0$ denotes the mass density per unit volume within the i-th layer.

We find

$$\mathcal{K}_i = \frac{1}{2}\int_\Omega \rho_i h_i(\dot{w})^2 + \frac{\rho_i h_i^3}{12}\left(\dot{\psi}^i \cdot \dot{\psi}^i\right) + \rho_i h_i\left(\dot{v}^i \cdot \dot{v}^i\right) d\underline{x},$$

where the "dot product" denotes the usual scalar product on $\mathbb{R}^2$.

2.4 Work. To set ideas we will assume that the plate is clamped on a portion of its edge $\Gamma_0 \subset \Gamma$ of positive measure. Furthermore denote $\Gamma_1 = \Gamma - \Gamma_0$.

Now assume the composite plate is subject to a volume distribution of forces $(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ and a distribution of forces $(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)$ along Γ_1. The work done on the plate by these forces is $\mathcal{W} = \sum_{i=1}^n \mathcal{W}_i$, where

$$\mathcal{W}_i = \int_{Q_i} \sum_{j=1,2,3} \tilde{f}_j U_j \, d\underline{x} dx_3 + \int_{z_{i-1}}^{z_i}\int_{\Gamma_1} \sum_{j=1,2,3} \tilde{g}_j U_j \, d\Gamma dx_3.$$

For $\underline{x} \in \Omega$ and $s \in \Gamma_1$ denote

$$f_3(\underline{x}) = \int_0^h \tilde{f}_3(\underline{x}, x_3)\, dx_3, \qquad g_3(s) = \int_0^h \tilde{g}_3(s, x_3)\, dx_3, \tag{2.9}$$

and for $(i = 1, 2, \dots, n)$,

$$\begin{cases} f_j^i(\underline{x}) = \int_{z_{i-1}}^{z_i} \tilde{f}_j dx_3 & M_j^i(\underline{x}) = \int_{z_{i-1}}^{z_i} (x_3 - \hat{z}_i)\tilde{f}_j dx_3, \quad j = 1, 2 \\ g_j^i(s) = \int_{z_{i-1}}^{z_i} \tilde{g}_j^i dx_3 & m_j^i(s) = \int_{z_{i-1}}^{z_i} (x_3 - \hat{z}_i)\tilde{g}_j^i dx_3 \quad j = 1, 2. \end{cases} \tag{2.10}$$

Furthermore, for $i = 1, 2, \dots$ denote $f^i = \{f_1^i, f_2^i\}$, $M^i = \{M_1^i, M_2^i\}$, and likewise for g^i and m^i. Due to the assumptions in (2.7), the expression for the work can be expressed in terms of the resultants in (2.9)–(2.10) and we obtain

$$\mathcal{W} = \int_\Omega w f_3 + \sum_{i=1}^n (\psi^i \cdot M^i + v^i \cdot f^i) d\underline{x} + \int_{\Gamma_1} w g_3 + \sum_{i=1}^n (\psi^i \cdot m^i + v^i \cdot g^i) d\Gamma. \tag{2.11}$$

We will also need to refer to the force resultants along the interfaces. If in place of (2.10) we define $p_j^i(\underline{x})$ and $q_j^i(s)$, $i = 0, 1, 2, \dots, n$, $j = 1, 2$ to be the effective resultant forces acting at the points $\{\underline{x}, z_i\}$ ($\underline{x} \in \Omega$) and $\{s, z_i\}$ ($s \in \Gamma_1$) then (2.11) takes the form

$$\mathcal{W} = \int_\Omega w f_3 + \sum_{i=1}^n (u^i \cdot p^i)\, d\underline{x} + \int_{\Gamma_1} w g_3 + \sum_{i=1}^n (u^i \cdot q^i)\, d\Gamma. \tag{2.12}$$

These resultant forces used in (2.12) are related to those in (2.10) by

$$\begin{cases} p^0 & = f^1/2 - M^1/h_1 \\ p^i & = (f^i + f^{i+i})/2 + M^i/h_i - M^{i+1}/h_{i+1} \qquad i = 1, 2, \dots n-1 \\ p^n & = f^n/2 + M^n/h_n \end{cases} \tag{2.13}$$

and the q^i are related to g^i and m^i in the same way.

3. Equations of motion. The Lagrangian $\mathcal{L}$ on $(0, T)$ is defined by

$$\mathcal{L} = \int_0^T \mathcal{K}(t) + \mathcal{W}(t) - \mathcal{P}(t) dt.$$

According to the principle of virtual work, the solution trajectory is the trajectory which renders stationary the Lagrangian under all kinematically admissible displacements. In this section we use this approach to derive the weak form of the equations of motion and then determine the associated boundary value problem.

First however, some additional notation is called for.

3.1. Notation. Let us first define the following n by n matrices:

$$\mathbf{h} = \operatorname{diag}(h_1, h_2, \dots, h_n) \qquad \mathbf{D} = \operatorname{diag}(D_1, D_2, \dots, D_n)$$
$$\rho = \operatorname{diag}(\rho_1, \rho_2, \dots, \rho_n) \qquad \mathbf{G} = \operatorname{diag}(G_1, G_2, \dots, G_n).$$

Let ψ, φ, v, f, g, M, and m be the n by 2 matrices defined by

$$(\psi)_{ij} = \psi_j^i \qquad i = 1, 2, \dots, n, \ j = 1, 2,$$

and so forth for φ, v, f, g, M and m. Also let u, p, q denote the $n+1$ by 2 matrices defined by

$$(u)_{ij} = u_j^i \qquad i = 0, 1, 2, \dots, n, \ j = 1, 2,$$

and likewise for p and q (p_j^i and q_j^i are defined by (2.13)).

If we let S^+ and S^- denote the n by $n+1$ matrices defined by

$$S^+\{y_0, y_1, \dots, y_n\}^T = \{y_1, y_2, \dots, y_n\}^T$$
$$S^-\{y_0, y_1, \dots, y_n)^T = (y_0, y_1, \dots, y_{n-1}\}^T$$

then ψ and v are given in terms of u as

$$\psi = \mathbf{h}^{-1}(S^+ - S^-)u \qquad v = 2^{-1}(S^+ + S^-)u, \tag{3.1}$$

and φ is given in terms of u and w by

$$\varphi = \psi + \{\nabla w, \nabla w, \dots, \nabla w\}^T \triangleq \psi + \vec{\nabla} w, \tag{3.2}$$

where ∇w repeats n times.

If θ and ξ are matrices in $\mathbb{R}^{lm}$, by $\theta \cdot \xi$ we mean the scalar product in $\mathbb{R}^{lm}$. We will also denote

$$(\theta, \xi)_\Omega = \int_\Omega \theta \cdot \xi \, d\underline{x}, \qquad (\theta, \xi)_{\Gamma_1} = \int_{\Gamma_1} \theta \cdot \xi \, d\Gamma.$$

3.2 Weak form of equations of motion. The expressions for the kinetic and potential energy and work can be rewritten as

$$\begin{aligned}
\mathcal{K}(t) &= \tilde{c}(\dot{v},\dot{\psi},\dot{w};\dot{v},\dot{\psi},\dot{w})/2\\
\mathcal{P}(t) &= \tilde{a}(v,\psi,\varphi;v,\psi,\varphi)/2\\
\mathcal{W}(t) &= (\psi,M)_\Omega + (v,f)_\Omega + (w,f_3)_\Omega + (\psi,m)_{\Gamma_1} + (v,g)_{\Gamma_1} + (w,g_3)_{\Gamma_1},
\end{aligned}$$

where $\tilde{c}(\cdot\,;\cdot)$ and $\tilde{a}(\cdot\,;\cdot)$ denote the bilinear forms

$$\begin{aligned}
\tilde{c}(\psi,v,w;\hat{\psi},\hat{v},\hat{w}) &= ((\mathbf{h}\cdot\rho)w,\hat{w})_\Omega + ((\rho\mathbf{h}^3/12)\psi,\hat{\psi})_\Omega + (\mathbf{h}\rho v,\hat{v})_\Omega\\
\tilde{a}(\psi,v,\varphi;\hat{\psi},\hat{v},\hat{\varphi}) &= \tilde{a}_0(\mathbf{h}^3\psi;\hat{\psi}) + 12\tilde{a}_0(\mathbf{h}v;\hat{v}) + \tilde{a}_1(\varphi;\hat{\varphi})\\
\tilde{a}_0(\psi,\hat{\psi}) &= \sum_{i=1}^{n}\tilde{a}_0^i(\psi^i;\hat{\psi}^i)\\
\tilde{a}_0^i(\psi^i;\hat{\psi}^i) &= \left(D_i\frac{\partial\psi_1^i}{\partial x_1},\frac{\partial\hat{\psi}_1^i}{\partial x_1}\right)_\Omega + \left(D_i\frac{\partial\psi_2^i}{\partial x_2},\frac{\partial\hat{\psi}_2^i}{\partial x_2}\right)_\Omega\\
&\quad + \left(\nu_i D_i\frac{\partial\psi_2^i}{\partial x_2},\frac{\partial\hat{\psi}_1^i}{\partial x_1}\right)_\Omega + \left(\nu_i D_i\frac{\partial\psi_1^i}{\partial x_1},\frac{\partial\hat{\psi}_2^i}{\partial x_2}\right)_\Omega\\
&\quad + \left(\left(\frac{1-\nu_i}{2}\right)D_i\left(\frac{\partial\psi_1^i}{\partial x_2}+\frac{\partial\psi_2^i}{\partial x_1}\right),\left(\frac{\partial\hat{\psi}_1^i}{\partial x_2}+\frac{\partial\hat{\psi}_2^i}{\partial x_1}\right)\right)_\Omega\\
\tilde{a}_1(\varphi;\hat{\varphi}) &= (\mathbf{G}\mathbf{h}\varphi,\hat{\varphi})_\Omega.
\end{aligned}$$

Let $\{\hat{u},\hat{w}\} = \{\hat{u}^0,\hat{u}^1,\ldots,\hat{u}^n,\hat{w}\}$ denote a test function on $\Omega\times(0,T)$ (with dimensionality matching that of $\{u,w\}$) for which

$$\begin{aligned}
\{\hat{u},\hat{w}\} = \{\frac{\partial\hat{u}}{\partial n},\frac{\partial\hat{w}}{\partial n}\} = 0 \quad \text{on } \Gamma_0\times(0,T)\\
\{\hat{u},\hat{w}\}|_{t=0} = \frac{\partial}{\partial t}\{\hat{u},\hat{w}\}|_{t=0} = \{\hat{u},\hat{w}\}|_{t=T} = \frac{\partial}{\partial t}\{\hat{u},\hat{w}\}|_{t=T} = 0 \quad \text{in } \Omega
\end{aligned} \tag{3.3}$$

where n is the outward unit normal to Γ. We set

$$0 = \langle\mathcal{L}'(u,w),(\hat{u},\hat{w})\rangle = \lim_{\epsilon\to0}\frac{\mathcal{L}\left((u,w)+\epsilon(\hat{u},\hat{w})\right)-\mathcal{L}(u,w)}{\epsilon}$$

to obtain the equations of motion in weak form:

$$\begin{aligned}
\int_0^T \tilde{c}\left(\dot{\psi},\dot{v},\dot{w};\dot{\hat{\psi}},\dot{\hat{v}},\dot{\hat{w}}\right) - \tilde{a}\left(\psi,v,\varphi;\hat{\psi},\hat{v},\hat{\varphi}\right) + (f_3,\hat{w})_\Omega\\
+ (M,\hat{\psi})_\Omega + (f,\hat{v})_\Omega + (m,\hat{\psi})_{\Gamma_1} + (g,\hat{v})_{\Gamma_1} + (g_3,\hat{w})_{\Gamma_1}\,dt = 0
\end{aligned} \tag{3.4}$$

where $\hat{v}$, $\hat{\psi}$ and $\hat{\varphi}$ are given in terms of $\{\hat{u}, \hat{w}\}$ by (3.1), (3.2) (but with hats on u, v, w, φ, ψ).

Define the forms a and c by

$$a(u, w; \hat{u}, \hat{w}) = \tilde{a}(\psi, v, \varphi; \hat{\psi}, \hat{v}, \hat{\varphi}) \tag{3.5}$$

$$c(u, w; \hat{u}, \hat{w}) = \tilde{c}(\psi, v, w; \hat{\psi}, \hat{v}, \hat{w}) \tag{3.6}$$

where $\{\psi, v, \varphi\}$ are related to $\{u, w\}$ by (3.1), (3.2) and $\{\hat{\psi}, \hat{v}, \hat{\varphi}\}$ are likewise related to $\{\hat{u}, \hat{v}\}$.

From (2.13) p and q can be rewritten as

$$\begin{cases} p &= (S^+ - S^-)^T \mathbf{h}^{-1} M + (S^+ + S^-)^T f/2 \\ q &= (S^+ - S^-)^T \mathbf{h}^{-1} m + (S^+ + S^-)^T g/2. \end{cases} \tag{3.7}$$

In terms of the original variables u, w and test functions in (3.3), it follows from (3.1), (3.7) that the equation of motion (3.4) may be rewritten

$$\begin{aligned} &\int_0^T c\left(\dot{u}, \dot{w}; \dot{\hat{u}}, \dot{\hat{w}}\right) - a\left(u, w; \hat{u}, \hat{w}\right) \\ &+ (f_3, \hat{w})_\Omega + (p, \hat{u})_\Omega + (g_3, \hat{w})_{\Gamma_1} + (q, \hat{u})_{\Gamma_1}\, dt = 0. \end{aligned} \tag{3.8}$$

3.3 Associated boundary value problem. The boundary value problem associated with (3.8) (or equivalently (3.4)) can be obtained through the usual integration by parts procedure.

Define for $i = 1, 2, \ldots n$ and sufficiently smooth $\phi = \{\phi_1, \phi_2\}(\underline{x})$ the operators $\mathcal{M}^i[\phi]$ by

$$\mathcal{M}^i[\phi] = D_i \begin{pmatrix} \varepsilon_{11} + \nu_i \varepsilon_{22} & (1-\nu_i)\varepsilon_{12} \\ (1-\nu_i)\varepsilon_{12} & \nu_i \varepsilon_{11} + \varepsilon_{22} \end{pmatrix}; \quad \varepsilon_{jk}(\phi) = \frac{1}{2}\left(\frac{\partial \phi_j}{\partial x_k} + \frac{\partial \phi_k}{\partial x_j}\right).$$

Thus $\mathcal{M}^i[\phi]$ is a symmetric matrix for every $\underline{x} \in \Omega$.

We define the *div* of a symmetric matrix to be the divergence of each row:

$$div \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} = \{\mathrm{div}(a_{11}, a_{12}), \mathrm{div}(a_{12}, a_{22})\}.$$

Then

$$L^i\phi = \{L^i_1\phi, L^i_2\phi\} = div\ \mathcal{M}^i[\phi]$$

defines a second order operator which is given explicitly by

$$\begin{aligned} L^i_j\phi = &\frac{\partial}{\partial x_j}\left(D_i \frac{\partial \phi_j}{\partial x_j}\right) + \frac{\partial}{\partial x_k}\left(\frac{(1-\nu_i)}{2} D_i \frac{\partial \phi_j}{\partial x_k}\right) + \frac{\partial}{\partial x_k}\left(\frac{(1-\nu_i)}{2} D_i \frac{\partial \phi_k}{\partial x_j}\right) \\ &+ \frac{\partial}{\partial x_j}\left(\nu_i D_i \frac{\partial \phi_k}{\partial x_k}\right), \quad (j,k) = (1,2),\ (2,1). \end{aligned}$$

Let us also define the boundary operators $\mathcal{B}^i\phi = \{\mathcal{B}_1^i(\phi_1,\phi_2),(\mathcal{B}_2^i(\phi_1,\phi_2)\}$ by

$$\mathcal{B}^i\phi = \mathcal{M}^i[\phi]n,$$

where $n=(n_1,n_2)$ denotes the outward unit normal to Γ. Explicitly one has

$$\mathcal{B}_1^i(\phi_1,\phi_2) = D_i\left[\left(\frac{\partial\phi_1}{\partial x_1}n_1 + \nu_i\frac{\partial\phi_2}{\partial x_2}n_1\right) + \left(\frac{1-\nu_i}{2}\right)\left(\frac{\partial\phi_1}{\partial x_2}+\frac{\partial\phi_2}{\partial x_1}\right)n_2\right]$$
$$\mathcal{B}_2^i(\phi_1,\phi_2) = D_i\left[\left(\frac{\partial\phi_2}{\partial x_2}n_2 + \nu_i\frac{\partial\phi_1}{\partial x_1}n_2\right) + \left(\frac{1-\nu_i}{2}\right)\left(\frac{\partial\phi_2}{\partial x_1}+\frac{\partial\phi_1}{\partial x_2}\right)n_1\right].$$

Assuming that the coefficients D_i, μ_i and G_i $(i=1,\dots n)$ are smooth enough to allow integration by parts, the following Green's formula is valid for all sufficiently smooth $\hat\phi$, ϕ:

$$\tilde a_0^i(\phi,\hat\phi) = (\mathcal{B}^i\phi,\hat\phi)_\Gamma - (L^i\phi,\hat\phi)_\Omega. \tag{3.9}$$

For $\xi = (\xi_j^i)$ $(i=1,2,\dots,n,\ j=1,2)$ define the matrices $L\xi$ and $\mathcal{B}\xi$ by

$$(L\xi)_{ij} = (L_j^i\xi^i), \qquad (\mathcal{B}\xi)_{ij} = (\mathcal{B}_j^i\xi^i), \qquad i=1,2,\dots,n,\ j=1,2.$$

An integration by parts in t of (3.4) followed by an application of (3.9) leads to the following:

$$\begin{aligned} 0 = \int_\Omega &\hat w\left((\mathbf{h}\cdot\rho)\ddot w - \operatorname{div}(\sum_{i=1}^n G_i h_i\varphi^i) - f_3\right) + \hat v\cdot(\mathbf{h}\rho\ddot v - 12\mathbf{h}Lv - f) \\ &+ \hat\psi\cdot\left(\rho\mathbf{h}^3\ddot\psi/12 - \mathbf{h}^3L\psi + \mathbf{G}\mathbf{h}\varphi - M\right)\,d\underline{x} \\ + \int_{\Gamma_1} &\hat\psi\cdot(\mathbf{h}^3\mathcal{B}\psi - m) + \hat w\left((\sum_{i=1}^n G_i h_i\varphi^i)\cdot n - g_3\right) \\ &+ \hat v\cdot(12\mathbf{h}\mathcal{B}v - g)\,d\Gamma. \end{aligned} \tag{3.10}$$

Next, one must choose an appropriate set of state variables. (Note: $\{\psi,v,w\}$ are not appropriate since they are not independent variables.) If we use $\{u,w\}$ we obtain the following.

$$\begin{cases} \text{(i)} & (\rho\cdot\mathbf{h})\ddot w - \operatorname{div}(\sum_{i=1}^n G_i h_i\varphi^i) = f_3 \\ \text{(ii)} & (S^+ - S^-)^T\left[\rho\mathbf{h}^2\ddot\psi/12 + \mathbf{G}\varphi - \mathbf{h}^2L\psi\right] \\ & \quad +\frac{1}{2}(S^+ + S^-)^T\left[\mathbf{h}\rho\ddot v - 12\mathbf{h}Lv\right] = p \qquad \text{in } \Omega\times\mathbb{R}^+ \end{cases} \tag{3.11}$$

where ψ, φ and v are related to u by (3.1), (3.2). The boundary conditions are:

$$\begin{cases} \text{(i)} & w = 0 \\ \text{(ii)} & u = 0 \qquad \text{on } \Gamma_0\times\mathbb{R}^+, \end{cases} \tag{3.12}$$

$$\begin{cases} \text{(i)} & \sum_{k=i}^{n}(G_i h_i \varphi^i)\cdot n = g_3 \\ \text{(ii)} & (S^+ - S^-)^T \mathbf{h}^2 \mathcal{B}\psi + 6(S^+ + S^-)^T \mathbf{h}\mathcal{B}v = q \qquad \text{on } \Gamma_1 \times \mathbb{R}^+. \end{cases} \tag{3.13}$$

Initial conditions can be given as

$$\begin{cases} \text{(i)} & w(\underline{x},0) = \mathrm{w}^0(\underline{x}), \quad \dot{w}(\underline{x},0) = \mathrm{w}^1(\underline{x}) \\ \text{(ii)} & u(\underline{x},0) = \mathrm{u}^0(\underline{x}), \quad \dot{u}(\underline{x},0) = \mathrm{u}^1(\underline{x}) \qquad \underline{x} \in \Omega \end{cases} \tag{3.14}$$

for appropriate u^0, u^1, w^0, w^1.

3.4. Damped multilayer plates. Damping may be introduced into any of the plate layers by replacing the stress-strain relation (2.2) by an appropriate dissipative constitutive law. In the case of *strain rate damping*, the stresses depend not only on the strains, but also the strain rate and (2.4), (2.5) are modified to

$$\begin{cases} \sigma_{11} = \frac{E+\tilde{E}\,d/dt}{1-\nu^2}(\epsilon_{11} + \nu\epsilon_{22}) & \sigma_{12} = \frac{E+\tilde{E}\,d/dt}{1+\nu}\epsilon_{12} \\ \sigma_{22} = \frac{E+\tilde{E}\,d/dt}{1-\nu^2}(\nu\epsilon_{11} + \epsilon_{22}) & \sigma_{13} = 2(G + \tilde{G}\frac{d}{dt})\epsilon_{13} \\ \sigma_{33} = 0 & \sigma_{23} = 2(G + \tilde{G}\frac{d}{dt})\epsilon_{23}. \end{cases}$$

where $\tilde{E}$ and $\tilde{G}$ may depend upon $\underline{x}$ and i and are assumed to be nonnegative.

By the correspondence principle, the equations of motion are given by simply replacing E and G by $E + \tilde{E}d/dt$ and $G + \tilde{G}d/dt$, respectively. The equation of motion (3.8) is modified to

$$\begin{aligned} &\int_0^T c\left(\dot{u},\dot{w};\dot{\hat{u}},\dot{\hat{w}}\right) - b\left(\dot{u},\dot{w};\hat{u},\hat{w}\right) - a\left(u,w;\hat{u},\hat{w}\right) \\ &\quad + (f_3,\hat{w})_\Omega + (p,\hat{u})_\Omega + (g_3,\hat{w})_{\Gamma_1} + (q,\hat{u})_{\Gamma_1}\,dt = 0, \end{aligned} \tag{3.15}$$

where the form $b(\cdot\,;\cdot)$ is defined in an identical fashion as the way $a(\cdot\,;\cdot)$ was defined, however, with E and G replaced by $\tilde{E}$ and $\tilde{G}$. (Of course, since $\mathbf{D}$ and $\mathbf{G}$ are defined in terms of E and G, one also has to replace $\mathbf{D}$ and $\mathbf{G}$ by appropriate matrices, say $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{G}}$, that are defined accordingly.)

From this correspondence it is simple to write down the boundary value problem analogous to (3.11)-(3.13) when strain rate damping is included. For example, the equations in (3.11) become

$$\begin{cases} \text{(i)} & (\rho\cdot\mathbf{h})\ddot{w} - \operatorname{div}(\sum_{i=1}^{n} G_i h_i \varphi^i) - \operatorname{div}(\sum_{i=1}^{n} \tilde{G}_i h_i \dot{\varphi}^i) = f_3 \\ \text{(ii)} & (S^+ - S^-)^T\left[\rho\mathbf{h}^2\ddot{\psi}/12 + \mathbf{G}\varphi + \tilde{\mathbf{G}}\dot{\varphi} - \mathbf{h}^2 L\psi - \mathbf{h}^2\tilde{L}\dot{\psi}\right] \\ & \quad +\frac{1}{2}(S^+ + S^-)^T\left[\mathbf{h}\rho\ddot{v} - 12\mathbf{h}Lv - 12\mathbf{h}\tilde{L}\dot{v}\right] = p, \qquad \text{in } \Omega\times\mathbb{R}^+ \end{cases} \tag{3.16}$$

where $\tilde{L}$ is defined in the same way as L, but with $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{G}}$ in place of $\mathbf{D}$ and $\mathbf{G}$. The boundary conditions in (3.13) are modified in the same way.

For a discussion of general viscoelastic damping within the Reissner-Mindlin framework we refer the reader to [6].

4. Existence, uniqueness, regularity. In this section we discuss existence, uniqueness and regularity properties associated with solutions of the general damped (or undamped) plate equation (3.15).

Denote

$$H^1_{\Gamma_0} = \{\varphi : \varphi \in H^1(\Omega), \varphi = 0 \text{ on } \Gamma_0\}.$$

In the case that $\Gamma = \Gamma_0$, $H^1_{\Gamma_0} = H^1_0(\Omega)$. Also define

$$\mathcal{V} = (H^1_{\Gamma_0})^{2n+3}, \qquad \mathcal{H} = (L^2(\Omega))^{2n+3}.$$

The forms $a(\cdot\,;\cdot)$ and $c(\cdot\,;\cdot)$ defined in (3.5), (3.6) and the form $b(\cdot\,;\cdot)$ in (3.15) are symmetric and bilinear. For convenience, denote by $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ the corresponding (nonnegative) quadratic functions, e.g., $a(u,w) = a(u,w;u,w)$.

In all that follows we assume without further mention that the coefficients ρ_i, G_i, D_i, ν_i, etc., $(i = 1, \dots n)$ are bounded and measurable on Ω.

4.1. Homogeneous boundary conditions. We first consider the problem (3.15) in the absence of boundary forces. In this case a variational formulation of (3.15) is: Find functions $\{u, w\}$ such that

$$\{u,w\} \in C\left([0,T];\mathcal{V}\right) \cap C^1\left([0,T];\mathcal{H}\right), \tag{4.1}$$

$$\begin{aligned} &\frac{d}{dt}[c(\dot u, \dot w; \hat u, \hat w) + b(u,w;\hat u,\hat w)] + a(u,w;\hat u,\hat w) \\ &\quad = (f_3, \hat w)_\Omega + (p, \hat u)_\Omega \qquad \forall\,\{\hat u, \hat w\} \in \mathcal{V} \end{aligned} \tag{4.2}$$

in the sense of distributions on $(0,T)$, with

$$\begin{cases} \{u(0), w(0)\} = \{\mathrm{u}^0, \mathrm{w}^0\} & \text{given in } \mathcal{V} \\ \{\dot u(0), \dot w(0)\} = \{\mathrm{u}^1, \mathrm{w}^1\} & \text{given in } \mathcal{H}. \end{cases} \tag{4.3}$$

The forces $\{p, f_3\}$ are assumed to be in the class $L^2(0,T;\mathcal{H})$.

Lemma 4.1. *The forms $a(\cdot\,;\cdot)$, $b(\cdot\,;\cdot)$ and $c(\cdot\,;\cdot)$ are continuous on $\mathcal{V}\times\mathcal{V}$, $\mathcal{V}\times\mathcal{V}$ and $\mathcal{H}\times\mathcal{H}$, respectively. Furthermore there exist $\delta_a > 0$ and $\delta_c > 0$ for which*

$$a(u,w) > \delta_a \|\{u,w\}\|^2_{\mathcal{V}} \qquad \forall\{u,w\} \in \mathcal{V} \tag{4.4}$$

$$b(u,w) \geq 0 \qquad \forall\{u,w\} \in \mathcal{V} \tag{4.5}$$

$$c(u,w) > \delta_c \|\{u,w\}\|^2_{\mathcal{H}} \qquad \forall\{u,w\} \in \mathcal{H}. \tag{4.6}$$

Proof. A proof for the case $n = 1$ appears in [6, pp. 44–47] and the same idea remains valid here.

First note that $c(\cdot\,;\cdot)$ is continuous on $\mathcal{H}\times\mathcal{H}$ since it is a composition of continuous functions. Likewise $a(\cdot\,;\cdot)$ and $b(\cdot\,;\cdot)$ are continuous on $\mathcal{V}\times\mathcal{V}$. Since $b(\cdot)$ is clearly nonnegative, (4.5) holds.

In the following C_g represents a generic positive constant that may vary from line to line.

Let us prove (4.6). By the equivalence of finite dimensional norms, for any $\{u,w\}\in\mathcal{H}$,

$$\begin{aligned} c(u,w) &= \tilde{c}(\psi,v,w) \\ &\geq \mathrm{C}_g\left[(\psi,\psi)_\Omega + (v,v)_\Omega + (w,w)_\Omega\right] \\ &= \mathrm{C}_g\left[(\mathbf{h}^{-1}(S^+-S^-)u,\mathbf{h}^{-1}(S^+-S^-)u)_\Omega\right. \\ &\quad \left. + (1/4)((S^++S^-)u,(S^++S^-)u)_\Omega + (w,w)_\Omega\right] \\ &\geq \mathrm{C}_g\left[((S^+-S^-)u,(S^+-S^-)u)_\Omega\right. \\ &\quad \left. + ((S^++S^-)u,(S^++S^-)u)_\Omega + (w,w)_\Omega\right]. \end{aligned} \tag{4.7}$$

Next note that

$$([(S^+-S^-)^T(S^+-S^-) + (S^++S^-)^T(S^++S^-)]u,u)_\Omega \geq \mathrm{C}_g\|u\|_\Omega^2 \tag{4.8}$$

since otherwise (being a sum of two non-negative forms) S^++S^- and S^+-S^- have a common null-vector. This would imply the existence of $u\neq 0$ such that $S^+u=S^-u=0$ and this is impossible. Combining (4.7) and (4.8) gives (4.6).

Finally, we need to prove (4.4). It is shown in [6] that

$$\tilde{a}_0^i(\phi,\phi) \geq \mathrm{C}_g\|\phi\|^2_{(H^1(\Omega))^2} \qquad \forall\phi=\{\phi_1,\phi_2\}\in(H^1_{\Gamma_0})^2. \tag{4.9}$$

(The proof of (4.9) involves the use of Poincaré's inequality and Korn's inequality.)

By virtue of (4.9) we have

$$\tilde{a}_0(\phi,\phi) \geq \mathrm{C}_g\|\phi\|^2_{(H^1(\Omega))^{2n}}.$$

Consequently, in the same way that (4.7) and (4.8) were obtained we have

$$\begin{aligned} \tilde{a}_0(\mathbf{h}\psi,\psi) + \tilde{a}_0(\mathbf{h}v;v) &\geq \mathrm{C}_g\left(\|\psi\|^2_{(H^1(\Omega)^{2n}} + \|v\|^2_{(H^1(\Omega)^{2n}}\right) \\ &\geq \mathrm{C}_g\|u\|^2_{(H^1(\Omega))^{2n+2}}. \end{aligned} \tag{4.10}$$

It follows from Poincaré's inequality that for any $\alpha>0$ and any $\{\phi,w\}\in(H^1_{\Gamma_0})^2\times H^1_{\Gamma_0}$ that

$$\|\phi\|^2_{(H^1(\Omega))^2} + \alpha(\nabla w+\phi,\nabla w+\phi)_\Omega \geq \mathrm{C}_g\|\{\phi,w\}\|^2_{(H^1(\Omega))^3}$$

(see [6, p. 44] for details). Consequently, for any $\{\phi, w\} \in (H^1_{\Gamma_0})^{2n} \times H^1_{\Gamma_0}$ it follows that

$$\|\phi\|^2_{(H^1(\Omega))^{2n}} + \alpha(\vec{\nabla} w + \phi, \vec{\nabla} w + \phi)_\Omega \geq \mathrm{C}_g \|\{\phi, w\}\|^2_{(H^1(\Omega))^{2n+1}}. \tag{4.11}$$

Combining (4.10) with (4.11) (with ψ and v in place of ϕ) we obtain (4.4). □

With the coercive estimates in Lemma 1.1, it is rather standard to show that (4.1)–(4.3) is well-set. We briefly indicate the semigroup approach used in Lagnese [5, p.29], to which the reader is referred for details.

We identify $\mathcal{H}$ with its dual $\mathcal{H}'$ and have the dense and continuous embeddings

$$\mathcal{V} \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{V}'.$$

Let $< \cdot, \cdot >$ denote the duality pairing between $\mathcal{V}$ and $\mathcal{V}'$ which coincides with $(\cdot, \cdot)_\mathcal{H}$ when both arguments are in $\mathcal{H}$. We may define (by Lemma 4.1) operators $C \in \mathcal{L}(\mathcal{H})$, $B \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ and $A \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ by

$$\begin{aligned} < C\mathbf{u}, \mathbf{v} > &= c(\mathbf{u}; \mathbf{v}) \quad \forall\ \mathbf{u}, \mathbf{v} \in \mathcal{H} \\ < B\mathbf{u}, \mathbf{v} > &= b(\mathbf{u}; \mathbf{v}) \quad \forall\ \mathbf{u}, \mathbf{v} \in \mathcal{V} \\ < A\mathbf{u}, \mathbf{v} > &= a(\mathbf{u}; \mathbf{v}) \quad \forall\ \mathbf{u}, \mathbf{v} \in \mathcal{V}. \end{aligned}$$

By Lemma 4.1, A and C are isomorphisms: $\mathcal{V} \to \mathcal{V}'$ and $\mathcal{H} \to \mathcal{H}$, respectively.

We set

$$\mathbf{u} = \{u, w\}, \quad \mathbf{f} = \{p, f_3\}, \quad \mathbf{u}^0 = \{\mathrm{u}^0, \mathrm{w}^0\}, \quad \mathbf{u}^1 = \{\mathrm{u}^1, \mathrm{w}^1\}$$

and rewrite (4.2)–(4.3) as

$$C\ddot{\mathbf{u}} + B\dot{\mathbf{u}} + A\mathbf{u} = \mathbf{f} \quad \text{in } \mathcal{V}' \qquad \mathbf{u}(0) = \mathbf{u}^0,\ \dot{\mathbf{u}}(0) = \mathbf{u}^1. \tag{4.12}$$

In first order form (4.12) becomes

$$\frac{d}{dt}\mathcal{C}\mathbf{U} + \mathcal{A}\mathbf{U} = \mathbf{F}, \qquad \mathbf{U}(0) = \{\mathbf{u}^0, \mathbf{u}^1\}^T, \tag{4.13}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & -A \\ A & B \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 \\ \mathbf{f} \end{pmatrix}.$$

One can then verify that $\mathcal{C}^{-1}\mathcal{A}$ is the generator of a strongly continuous contraction semigroup on (for example) $\mathcal{V} \times \mathcal{H}$. As solutions in $\mathcal{V} \times \mathcal{H}$ can be shown to correspond to variational solutions of (4.1)–(4.3) we have the following result.

Proposition 4.2. *There is a unique solution* $\{u, w\}$ *to* (4.1)–(4.3)*. Moreover the mapping* $[\{u^0, w^0\}, \{u^1, w^1\}, \{p, f_3\}] \to [\{u, w\}, \{\dot{u}, \dot{w}\}]$ *is a continuous mapping of*

$$\mathcal{V} \times \mathcal{H} \times L^2(0, T; \mathcal{H}) \to C([0, T]; \mathcal{V} \times \mathcal{H}).$$

Remark 4.3. Weaker classes of solutions may also be defined by extension of the relevant semigroup. In particular, define $\mathcal{Z}$ to be the completion of $\mathcal{V} \times \mathcal{H}$ with respect to the norm

$$\|\{\mathbf{u}, \mathbf{v}\}\|_{\mathcal{Z}} = \|\mathcal{A}^{-1}\mathcal{C}\{\mathbf{u}, \mathbf{v}\}\|_{\mathcal{V}\times\mathcal{H}}.$$

It follows that

$$\mathcal{Z} = \{\ \{\mathbf{u}, \mathbf{v}\} \in \mathcal{H} \times (\mathcal{V}' + B\mathcal{H}) : B\mathbf{u} + \mathbf{v} \in \mathcal{V}'\}.$$

It is easily verified that semigroup generated by $\mathcal{C}^{-1}\mathcal{A}$ (used in Proposition 4.2) extends continuously to the space $\mathcal{Z}$ and hence given any $\{\mathbf{u}^0, \mathbf{u}^1\} \in \mathcal{Z}$ and any $\mathbf{F} \in L^2(0, T; \mathcal{Z})$ there is a unique solution $\mathbf{U}$ of (4.13) for which $\mathbf{U} \in C([0, T]; \mathcal{Z})$.

4.3. Nonhomogeneous boundary data. The presence of the boundary forces g_3 and q in (3.15) makes it impossible to directly define solutions by the variational approach. Instead, one must initially define a weaker notion of solution. In applying the method of *transposition* (see [7]), solutions are defined by duality with respect to solutions of an adjoint problem. This method is applied in the case of a single-layer Reissner-Mindlin plate in [6, p. 64,65] and the same approach is valid here. Therefore we simply state the result [6, Theorem 3.3, p. 65] as it applies to our problem.

Proposition 4.4. *Assume* $\{\{u^0, w^0\}, \{u^1, w^1\}\} \in \mathcal{Z}$*,* $\{p, f_3\} \in L^2(0, T; \mathcal{V}')$ *and* $\{q, g_3\} \in L^2(0, T; (H^{1/2}(\Gamma_1))^{2n+3})'$*. Then there exists a unique solution (in the sense of transposition)* $\{u, w\}$ *to* (3.15) *with*

$$\{\ \{u, w\}, \{\dot{u}, \dot{w}\ \}\ \} \in C([0, T]; \mathcal{Z}).$$

Moreover, the mapping of

$$[\ \{\ \{u^0, w^0\}, \{u^1, w^1\}\ \}, \{p, f_3\}, \{q, g_3\}\ ] \to \{\ \{u, w\}, \{\dot{u}, \dot{w}\}\}$$

is a continuous mapping of

$$\mathcal{Z} \times L^2(0, T; \mathcal{V}') \times L^2(0, T; (H^{1/2}(\Gamma_1))^{2n+3})' \to C([0, T]; \mathcal{Z}).$$

5. Symmetric case. In this section we examine the special case where the thicknesses, densities, and elastic parameters are symmetric with respect to the center layer (if n is odd) or the center interface (if n is even). We show that the equations of motion decouple into a *bending part* and a *stretching part*; that is, a part invloving the transverse displacement and another part completely independent of the transverse displacement. Solutions of the bending equations turn out to have antisymmetric displacements with respect to the center of the plate, while those of the stretching part have symmetric displacements.

5.1 Symmetry properties. Let R denote the matrix which reverses the coordinates in $\mathbb{R}^m$ (m arbitrary):

$$R\{\phi_1, \phi_2, \ldots, \phi_m\}^T = \{\phi_m, \phi_{m-1}, \ldots, \phi_1\}^T.$$

For any vector ϕ in $\mathbb{R}^m$ define the *symmetric part* ϕ_s and *antisymmetric part* ϕ_a by

$$\phi_s = \frac{\phi + R\phi}{2}, \qquad \phi_a = \frac{\phi - R\phi}{2}. \tag{5.1}$$

Thus by (5.1) every vector ϕ has the orthogonal decomposition

$$\phi = \phi_s + \phi_a, \qquad \phi_s \cdot \phi_a = 0. \tag{5.2}$$

In this section we assume that all physical parameters (E, G, ν, thicknesses, densities, etc.) are symmetric with respect to the layers. If Λ denotes any of the matrices ρ, $\mathbf{h}, \mathbf{D}, \tilde{\mathbf{D}}, \mathbf{G}, \tilde{\mathbf{G}}$, $\mathrm{diag}(\nu_1, \ldots, \nu_n)$ then it is easy to check that

$$R\Lambda = \Lambda R. \tag{5.3}$$

Recall the general variational problem: Find functions $\{u, w\}$ which satisfy

$$\{u, w\} \in C([0, T]; \mathcal{V}) \cap C^1([0, T]; \mathcal{H}), \tag{5.4}$$

$$\frac{d}{dt}[c(\dot{u}, \dot{w}; \hat{u}, \hat{w}) + b(u, w; \hat{u}, \hat{w})] + a(u, w; \hat{u}, \hat{w}) \tag{5.5}$$

$$= (f_3, \hat{w})_\Omega + (p, \hat{u})_\Omega + (q, \hat{u})_{\Gamma_1} + (g_3, \hat{w})_{\Gamma_1} \quad \forall \{\hat{u}, \hat{w}\} \in \mathcal{V}$$

$$\{u(0), w(0)\} = \{u^0, w^0\}, \qquad \{\dot{u}(0), \dot{v}(0)\} = \{u^1, v^1\}. \tag{5.6}$$

Theorem 5.1. *Suppose the thicknesses, densities and elastic parameters are symmetric with respect to the center of the plate, i.e,* (5.3) *holds. Then every solution* $\{u, w\}$ *of* (5.4)–(5.6) *can be decomposed as*

$$\{u, w\} = \{u_a, w\} + \{u_s, 0\}, \tag{5.7}$$

where $\{u_a, w\}$ *satisfies* (5.4) *and is a solution of*

$$\frac{d}{dt}[c(\dot{u}_a, \dot{w}; \hat{u}, \hat{w}) + b(u_a, w; \hat{u}, \hat{w})] + a(u_a, w; \hat{u}, \hat{w}) \tag{5.8}$$
$$= (f_3, \hat{w})_\Omega + (p_a, \hat{u})_\Omega + (q_a, \hat{u})_{\Gamma_1} + (g_3, \hat{w})_{\Gamma_1} \quad \forall \{\hat{u}, \hat{w}\} \in \mathcal{V}$$
$$\{u_a(0), w(0)\} = \{u_a^0, w^0\}, \qquad \{\dot{u}_a(0), \dot{w}(0)\} = \{u_a^1, w^1\}, \tag{5.9}$$

and $\{u_s, 0\}$ *satisfies* (5.4) *and is a solution of*

$$\frac{d}{dt}[c(\dot{u}_s, 0; \hat{u}, \hat{w}) + b(u_s, 0; \hat{u}, \hat{w})] + a(u_s, 0; \hat{u}, \hat{w}) \tag{5.10}$$
$$= (p_s, \hat{u})_\Omega + (q_s, \hat{u})_{\Gamma_1} \quad \forall \{\hat{u}, \hat{w}\} \in \mathcal{V}$$
$$u_s(0) = u_s^0, \qquad \dot{u}_s(0) = u_s^1. \tag{5.11}$$

Remark 5.2. The proof is completely algebraic and hence the same result holds for weaker classes of solutions. (In particular, those given in Proposition 4.4)

Proof. The continuity of $\{u_a, w\}$ and $\{u_s, 0\}$ is the same as $\{u, w\}$. Furthermore (5.9) and (5.11) follow immediately from (5.2). The main point is to verify the invariance in (5.8) and (5.10).

Suppose we prove that that for any $\{u, w\} \in \mathcal{V} \times \mathcal{H}$ and any $\{\hat{u}, \hat{w}\} \in \mathcal{V} \times \mathcal{H}$

$$a(u_a, w; \hat{u}_s, 0) = b(u_a, w; \hat{u}_s, 0) = c(u_a, w; \hat{u}_s, 0) = 0. \tag{5.12}$$

It then would follow that

$$a(u, w; \hat{u}, \hat{w}) = a(u_a, w; \hat{u}_a, \hat{w}) + a(u_s, 0; \hat{u}_s, 0)$$
$$b(u, w; \hat{u}, \hat{w}) = b(u_a, w; \hat{u}_a, \hat{w}) + b(u_s, 0; \hat{u}_s, 0)$$
$$c(u, w; \hat{u}, \hat{w}) = c(u_a, w; \hat{u}_a, \hat{w}) + c(u_s, 0; \hat{u}_s, 0).$$

Then, for example, (5.8) would be the same as

$$\frac{d}{dt}[c(\dot{u}_a, \dot{w}; \hat{u}_a, \hat{w}) + b(u_a, w; \hat{u}_a, \hat{w})] + a(u_a, w; \hat{u}_a, \hat{w}) = (f_3, \hat{w})_\Omega$$
$$+ (p_a, \hat{u}_a)_\Omega + (q_a, \hat{u}_a)_{\Gamma_1} + (g_3, \hat{w})_{\Gamma_1}, \qquad \forall \{\hat{u}, \hat{w}\} \in \mathcal{V},$$

which is precisely the equation obtained from (5.5) with $\hat{u}_s = 0$.

Therefore it will suffice to prove (5.12).

Let φ, ψ and v be defined by u, w by (3.1), (3.2) and assume $\hat{\varphi}$, $\hat{\psi}$, $\hat{v}$ are likewise defined in terms of $\hat{u}$, $\hat{w}$.

Let S^+ and S^- denote the matrices in (3.1). One can easily verify the following:

$$RS^+\phi = S^-R\phi, \qquad RS^-\phi = S^+R\phi, \qquad \forall\, \phi \in \mathbb{R}^n. \tag{5.13}$$

Of course the matrices R appearing on each side of the equations in (5.13) are not the same since their dimensions are different.

The following are easily obtained using (5.13).

$$\psi_s = \mathbf{h}^{-1}(S^+ - S^-)u_a, \qquad \psi_a = \mathbf{h}^{-1}(S^+ - S^-)u_s, \tag{5.14}$$

$$v_s = (S^+ + S^-)u_s/2 \qquad v_a = (S^+ + S^-)u_a/2 \tag{5.15}$$

$$\varphi_s = \vec{\nabla} w + \psi_s, \qquad \varphi_a = \psi_a. \tag{5.16}$$

For example, to obtain the first identity in (5.14), we use (5.3) (applied to $\mathbf{h}^{-1}$) and (5.13) to obtain

$$\begin{aligned}\psi_s &= (\mathbf{h}^{-1}(S^+ - S^-)u + R\mathbf{h}^{-1}(S^+ - S^-)u)/2\\ &= (\mathbf{h}^{-1}(S^+ - S^-)u + \mathbf{h}^{-1}(S^- - S^+)Ru)/2\\ &= \mathbf{h}^{-1}S^+(u - Ru)/2 - \mathbf{h}^{-1}S^-(u - Ru)/2 = \mathbf{h}^{-1}(S^+ - S^-)u_a.\end{aligned}$$

Equ. (5.16) follows from (5.14) and the fact that $R\vec{\nabla}w = \vec{\nabla}w$.

Now we can easily verify (5.12). For example,

$$\begin{aligned}c(u_a, w; \hat{u}_s, 0) &= \tilde{c}(\psi_s, v_a, w; \hat{\psi}_a, \hat{v}_s, 0)\\ &= \rho\mathbf{h}^3\psi_s \cdot \hat{\psi}_a + \rho\mathbf{h}v_a \cdot \hat{v}_s\\ &= R\rho\mathbf{h}^3\psi_s \cdot R\hat{\psi}_a + R\rho\mathbf{h}v_a \cdot R\hat{v}_s\\ &= \rho\mathbf{h}^3\psi_s \cdot (-1)\hat{\psi}_a + \rho\mathbf{h}(-1)v_a \cdot \hat{v}_s\\ &= -c(u_a, w; \hat{u}_s, 0),\end{aligned}$$

thus the last equality in (5.12) holds. The first two are proved in exactly the same way. □

5.2. Bending equations for symmetrically layered plates. It follows from the invariance of the antisymmetric solutions that the boundary value problem for the bending component of the motion is given by the the same sytem (3.11)–(3.14), with the forces and initial data replaced by their antisymmetric components, as in Theorem 5.1. However, the resulting equations are not independent due to the antisymmetry property of the solution. Roughly speaking, a minimal set of equations is given by the equations for the transverse displacement w together with the equations for the in-plane displacements of the "top half" of the plate.

Let us make this precise. Since the solutions are antisymmetric about center of the plate, the solution obtained for n even is identical to the solution obtained for n odd, once the center two layers are identified as a single layer. Thus we only need to consider the case where: n is even:

$$n = 2m.$$

Therefore assume that the layers of the plate are indexed $i = \pm 1, \pm 2, \ldots, \pm m$, and the surfaces are indexed $i = -m, \ldots, 0, \ldots, m$. Then by antisymmetry, $u_a^0 = 0$. Hence there are $2n+1$ equations; one for the transverse displacement and $2n$ for u_a.

Since $u_a^0 = 0$ the n equations for the upper half of the plate (those with $i > 0$) are decoupled from the n equations for the lower layers. Furthermore it is easily checked that the equations for the upper half are identical to the equations for the lower half, with the exception that the initial data and applied forces are opposite in sign. The boundary value problem for the upper half of the plate (without damping) then consists of (i) of (3.11)–(3.14) together with the first n equations (those involving only $i > 0$) in (ii) of (3.11)-(3.14). The in-plane displacements for the lower half of the plate are then determined by antisymmetry. The equations in the damped case are obtained in the same way.

As an example, let us write out the bending equations for a symmetric three-layered plate. Since the parameters $\rho, \mathbf{h}, \mathbf{G}, \mathbf{D}$ are symmetric, let us denote

$$\rho = \operatorname{diag}(\rho_1, \rho_0, \rho_1), \qquad \mathbf{h} = \operatorname{diag}(h_1, h_0, h_1), \quad \text{etc.} \tag{5.17}$$

Likewise the variables ψ and $\varphi = \vec{\nabla} w + \psi$ are symmetric while the variables v and u are antisymmetric:

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^0 \\ \psi^1 \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi^1 \\ \varphi^0 \\ \varphi^1 \end{pmatrix}, \quad v = \begin{pmatrix} v^1 \\ 0 \\ -v^1 \end{pmatrix}, \quad u = \begin{pmatrix} u^1 \\ u^0 \\ -u^0 \\ -u^1 \end{pmatrix}.$$

Since the external force p is also antisymmetric, assume p is indexed in the same way u is. For $\phi = \{\phi_1, \phi_2\}$ let us define

$$\square^i \phi = \frac{\rho_i h_i^2}{12} \ddot{\phi} - h_i^2 L^i \phi.$$

Then (3.11) becomes

$$\begin{array}{ll} \text{(i)} & (2\rho_1 h_1 + \rho_0 h_0)\ddot{w} - \operatorname{div}(2G_1 h_1 \varphi^1 + G_0 h_0 \varphi^0) = f_3 \\ \text{(ii)} & \begin{cases} \Box^1\psi^1 + G_1\varphi^1 + \frac{6}{h_1}\Box^1 v^1 = p^1 \\ \Box^0\psi^0 - \Box^1\psi^1 + G_0\varphi^0 - G_1\varphi^1 + \frac{6}{h_1}\Box^1 v^1 = p^0 \end{cases} \end{array} \tag{5.18}$$

which hold on $\Omega \times \mathbb{R}^+$. Recalling from (5.14)–(5.16) that

$$\psi^1 = \frac{u^1 - u^0}{h_1}, \quad \psi^0 = \frac{2u^0}{h_0}, \quad v^1 = \frac{u^1 + u^0}{2},$$

(ii) of (5.18) can be written in terms of u as

$$\text{(ii)} \qquad \begin{cases} \frac{1}{h_1}\Box^1(4u^1 + 2u^0) + G_1\varphi^1 = p^1 \\ \frac{1}{h_0}\Box^0(2u^0) + \frac{1}{h_1}\Box^1(2u^1 + 4u^0) + G_0(2u^0/h_0 + \nabla w) - G_1\varphi^1 = p^0 \\ \varphi^1 = \nabla w + (u^1 - u_0)/h_1. \end{cases}$$

If we use $\psi = \psi^1$ and $s = u^0/2h_1$ for state variables the system (5.18) can be written as

$$\begin{array}{ll} \text{(i)} & (2\rho_1 h_1 + \rho_0 h_0)\ddot{w} - \operatorname{div}(2G_1 h_1\varphi^1 + G_0 h_0 \nabla w + 4h_1 G_0 s) = f_3 \\ \text{(ii)} & 8h_1\Box^1(\psi + 3s) + (2h_1 G_1)\varphi^1 = 2h_1 p^1 \\ \text{(iii)} & 8h_1\Box^1 s + \frac{4}{3}\frac{(2h_1)^2}{h_0}(\Box^0 s + G_0 s) + \frac{4}{3}h_1 G_0 \nabla w - 2h_1 G_1 \varphi^1 = \tilde{M} \\ & \text{where} \qquad \varphi^1 = \psi + \nabla w, \qquad \tilde{M} = \frac{2h_1}{3}(2p^0 - p^1). \end{array} \tag{5.19}$$

Let us make a couple of purely formal observations (which can be made precise). As $h_0 \to 0$ we expect to recover the usual Reissner-Mindlin plate (i.e., the bending component of (3.11) with $n = 1$; see also [6, p.13]) since in the limit we are left with a symmetric two-layered plate, and as we have mentioned, this is the same as a one-layered plate once the two layers are identified as a single layer. Indeed, multiplying (iii) of (5.19) through by h_0 then formally setting h_0 to zero results in

$$\Box^0 s + G_0 s = 0. \tag{5.20}$$

If s and $\dot{s}$ are initially zero, and appropriate homogeneous boundary conditions are specified then the unique solution of (5.20) is $s \equiv 0$, as one would expect as $h_0 \to 0$. Next we set $s = 0$ and h_0 =0 in (i) and (ii), and replace $2h_1$ by h to obtain

$$\begin{array}{ll} \text{(i)} & \rho_1 h\ddot{w} - \operatorname{div}(G_1 h\varphi^1) = f_3 \\ \text{(ii)} & \frac{\rho_1 h^3}{12}\ddot{\psi} - h^3 L^1\psi + G_1 h\varphi^1 = hp^1 \\ & \text{where} \qquad \varphi^1 = \psi + \nabla w, \end{array}$$

which are precisely the Reissner-Mindlin equations for a single plate of thickness h, modulus of flexural rigidity $h^3 D_1$, modulus of shear hG_1 and density ρ_1.

We can also relate (5.19) to the plate model developed in Hansen [2], which describes a *two-layer plate with interfacial slip*. In this model, h_0 and G_0 are both assumed to be very small compared to $h = 2h_1$ and G_1. Thus in (5.19) we pass the limits $h_0 \to 0$ and $G_0 \to 0$ with $\gamma = \frac{G_0}{h_0}$ fixed. We obtain

$$\begin{aligned}
&\text{(i)} \qquad \rho_1 h\ddot{w} - \operatorname{div}(G_1 h\varphi^1) = f_3 \\
&\text{(ii)} \qquad \frac{\rho_1 h^3}{12}(\ddot{\psi} + 3\ddot{s}) - h^3 L^1(\psi + 3s) + hG_1\varphi^1 = hp^1 \\
&\text{(iii)} \qquad \frac{\rho_1 h^3}{12}\ddot{s} - h^3 L^1 s + \frac{4}{3}\gamma h^2 s - G_1 h\varphi^1 = \frac{h}{3}(2p^0 - p^1) \\
&\qquad\qquad \text{where} \qquad \varphi^1 = \psi + \nabla w,
\end{aligned}$$

which is precisely the system developed in [2].

5.3 Stretching equations of symmetrically layered plates. Likewise it follows from the invariance of the symmetric solutions that the boundary value problem for the stretching component of the motion (the part independent of w) is given by the the same system as for the general n-layer problem, when one replaces the forces and initial data by their symmetric components, as in Theorem 5.1.

Thus the boundary value problem (in the undamped case) is then the same as (3.11)–(3.14) but with the forces and initial data replaced by their antisymmetric components and with 0, ψ_a, v_s, ψ_a in place of w, ψ, v and φ, respectively.

The equations for the stretching component of the "upper half" of the plate again form an independent set of equations. To be more precise, assume for definiteness that n is even and the layers are indexed as in Section 5.2. Then the equations in (3.11)–(3.14) with $i > 0$ are identical to those with $i < 0$. Hence an independent set of equations for the middle interface and upper layers are given by those equations in (ii) of (3.11)–(3.14) with $i \geq 0$.

As an example, let us write out the stretching equations for a symmetric three-layered plate.

As in the example for the bending equations, we may assume that ρ, $\mathbf{D}$, $\mathbf{G}$, $\mathbf{h}$ are given by (5.17). This time ψ is an antisymmetric variable while v, u and p are symmetric. Thus we set

$$\psi = \begin{pmatrix} \psi^1 \\ 0 \\ -\psi^1 \end{pmatrix}, \quad v = \begin{pmatrix} v^1 \\ v^0 \\ v^1 \end{pmatrix}, \quad u = \begin{pmatrix} u^1 \\ u^0 \\ u^0 \\ u^1 \end{pmatrix} \quad p = \begin{pmatrix} p^1 \\ p^0 \\ p^0 \\ p^1 \end{pmatrix}.$$

Proceeding as discussed above, the stretching motions of (3.11) are

$$\begin{aligned} \Box^1\psi^1 + G_1\psi^1 + \frac{6}{h_1}\Box^1 v^1 &= p^1 \\ \frac{6}{h_0}\Box^0 v^0 + \frac{6}{h_1}\Box^1 v^1 - \Box^1\psi^1 - G^1\psi^1 &= p^0, \end{aligned} \tag{5.20}$$

which hold on $\Omega \times \mathbb{R}^+$. From (5.14)–(5.16) we have

$$v^0 = u^0, \qquad v^1 = \frac{u^1 + u^0}{2}, \qquad \psi^1 = \frac{u^1 - u^0}{h_1}.$$

Thus in terms of u (5.20) can be rewritten

$$\begin{aligned} \left(\frac{6}{h_0}\Box^0 + \frac{6}{h_1}\Box^1\right)u^0 + \frac{6}{h_1}\Box^1 u^1 &= p^0 + p^1 \\ -\left(\frac{\Box^1}{h_1} + \frac{3}{h_0}\Box^0 + \frac{G_1}{h_1}\right)u^0 + \frac{1}{h_1}(\Box^1 + G_1)u^1 &= (p_0 - p_1)/2. \end{aligned}$$

As $h_0 \to 0$, one sees (at least formally) that $\Box^0/h^0 \to 0$, so that in the limit one expects to obtain

$$\begin{aligned} \frac{6}{h_1}\Box^1(u^0 + u^1) &= p^0 + p^1 \\ \left(\frac{\Box^1}{h_1} + \frac{G_1}{h_1}\right)(u^1 - u^0) &= (p_0 - p_1)/2. \end{aligned} \tag{5.21}$$

The above are (as one would expect) the same equations we would have obtained assuming there were two symmetric layers instead of three. (To compare the equations, p_0 in (5.21) should be divided by two in order to have the same meaning as p_0 for the case $n = 2$.) Note that the two equations in (5.21) are completely decoupled. This is due to the fact that the wave speeds in each layer are the same and will not occur in general for $n \geq 3$. It is also worth noting that by letting $h = 2h_1$ and choosing v^1 for the state variable, the top equation of (5.21) takes the form

$$\rho_1 h\ddot{v}^1 - 12hL^i v^1 = 2(p^0 + p^1),$$

which, when the forces $2(p^0 + p^1)$ are interpreted as the resultant in-plane force, is the usual equation for plane elasticity (see also [6, p.17][1]).

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