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MODELING AND ANALYSIS OF A THREE-LAYER DAMPED SANDWICH BEAM

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Abstract. We follow a variational approach to derive the boundary value problem which models a constrained layer beam. When certain energy terms are ignored, this becomes a generalization of the well known Mead-Markus model [8]. We prove well-posedness on an appropriate energy state space. Then we study the effect of introducing damping into the middle layer of the structure, and determine the optimal damping as a function of the other material parameters.

1. Introduction.

A standard technique for vibration suppression in elastic structures is to make use of laminated members (beams/plates) which consist of a compliant middle layer sandwiched between two much stiffer outer layers. Then bending in the sandwich structure will induce large shear deformations in the compliant middle layer, and so energy can be dissipated via shear damping in this layer. This is often referred to as constrained layer damping, and we refer to [12] for references and discussion of constrained layer models.

In this paper we follow a variational approach to derive an 'accurate' model of a three-layer sandwich beam. By 'accurate' we mean that all relevant energy terms are included in the derivation of the partial differential equation model for the motion of the structure. We then neglect the effects of longitudinal and rotatory inertia to arrive at a model slightly more general than the well known Mead-Markus model. For this model we study properties of the solution semigroup. Then we introduce damping due to shear in the middle layer and, following on an idea in [5], we determine the optimal damping as a function of the other material parameters in the model.

2. The Model.

The constrained layer model under consideration consists of three beam layers, each of width r and length l. We take as standing assumptions that

- **(i):** no slip occurs along the interfaces,
- **(ii):** within each layer the longitudinal displacements vary linearly with respect to the transverse coordinate,
- **(iii):** within each layer the transverse displacement is constant with respect to the transverse coordinate, and
- **(iv):** the Kirchhoff hypothesis applies to the stiff outer layers, i.e. normal sections remain normal during displacement.

The base, core and constraining layers have thickness h_1, h_2 and h_3, respectively. Let

$$0 = z_0 < z_1 < z_2 < z_3 = h, \qquad h_i = z_i - z_{i-1},\ i = 1, 2, 3.$$

We use $\overline{x} = \{x_1, x_2, x_3\}$ to denote points in $Q = Q_1 \cup Q_2 \cup Q_3$, where

$$Q_i = (0, l) \times (-r/2, r/2) \times (z_{i-1}, z_i), \quad i = 1, 2, 3.$$

As a notational convention, we use index i to refer to beam/layer i. The index is superscripted for vector quantities which vary by layer, and subscripted for scalar quantities. Thus, at the equilibrium position, beam i occupies the region Q_i.

2.1. **Kinematic Assumptions.** For $\overline{x} \in Q$, let $\{U_1, U_2, U_3\}(\overline{x})$ denote the displacement vector of the point which has coordinate $\overline{x}$ when the beam is in equilibrium (for now we are suppressing the time variable t). In order to obtain a beam theory, all displacements are assumed to be independent of the x_2-coordinate, and deformations are zero in the x_2-direction. Thus, $U_2(\overline{x}) = 0$ for all $\overline{x} \in Q$. Also, because of assumption (iii) and the further assumption that the beams do not delaminate, it follows that

$$U_3(x_1, x_2, x_3) = U_3(x_1, \widetilde{x}_2, \widetilde{x}_3)$$

for all $(x_1, x_2, x_3), (x_1, \widetilde{x}_2, \widetilde{x}_3) \in Q$. In other words, the transverse displacement is a function of only the variable x_1, and so it can be denoted by $w(x_1)$. (To be precise, it is denoted by $w(x_1, t)$, but we are suppressing the time dependence at this stage of the discussion.) Let us next define

$$u^i(x_1) = U_1(x_1, 0, z_i),\ i = 1, 2, 3,\ \forall x_1 \in (0, l).$$

For $i = 1, 2, 3$, define ψ^i, φ^i, and v^i by

$$\psi^i = \frac{u^i - u^{i-1}}{h_i}, \quad \varphi^i = \psi^i + w_x, \quad v^i = \frac{u^{i-1} + u^i}{2}, \tag{1}$$

and set

$$\psi = [\psi^1\ \psi^2\ \psi^3]^T, \quad \varphi = [\varphi^1\ \varphi^2\ \varphi^3]^T, \quad v = [v^1\ v^2\ v^3]^T.$$

(Notational note: w_x denotes the partial derivative (partial because w is also a function of time t) with respect to the x_1 variable). The component ψ^i can be interpreted as the total rotation angle of the deformed filament within the ith layer in the x_1-x_3 plane. The components φ^i represent the (small angle approximation for the) shear angles within each layer. The component v^i represents the longitudinal displacement (stretching) within the ith layer.

It follows from assumption (iv) that $\varphi^1 = \varphi^3 = 0$ and $\psi^1 = \psi^3 = -w_x$. If we use the notation $\overline{1} = [1\ 1\ 1]^T$, then $\varphi = \psi + w_x\overline{1}$. Next let

$$\hat{z}_i = \frac{z_{i-1} + z_i}{2}$$

denote the centerline of the ith layer. Then

$$U_1(\overline{x}) = U_1(x_1, x_2, x_3) = v^i(x_1) + (x_3 - \hat{z}_i)\psi^i(x_1), \quad z_{i-1} < x_3 < z_i, \tag{2}$$

$$U_3(\overline{x}) = U_3(x_1, x_2, x_3) = w(x_1), \quad z_{i-1} < x_3 < z_i \tag{3}$$

Thus the displacement is uniquely determined in terms of the tranverse displacement w, the longitudinal displacement (stretching) v, and the total rotation angle ψ.

2.2. **Kinetic and Potential Energy.** The kinetic energy is the sum of the kinetic energy of each beam; $\mathcal{K} = \sum_{i=1}^{3} \mathcal{K}_i$, where

$$\mathcal{K}_i = \frac{1}{2} \int_{Q_i} \rho_i \left(\dot{U}_1^2 + \dot{U}_2^2 + \dot{U}_3^2 \right) d\overline{x}.$$

Here 'dot' represents differentiation with respect to time ('$\cdot$' $= \frac{\partial}{\partial t}$), and $\rho_i = \rho_i(\overline{x}) = \rho_i(x_1) > 0$ is the volume density in the ith layer. From the kinematic relations we find that

$$\mathcal{K}_i = r\frac{1}{2} \int_0^l \rho_i h_i (\dot{w})^2 + \frac{\rho_i h_i^3}{12} (\dot{\psi}^i)^2 + \rho_i h_i (\dot{v}_i)^2\, dx_1$$

Likewise the potential energy is the sum of the potential energy of each beam; $\mathcal{P} = \sum_{i=1}^{3} \mathcal{P}_i$, where

$$\mathcal{P}_i = \frac{1}{2}\int_{Q_i} \sum_{i,j=1}^{3} \sigma_{ij}\epsilon_{ij}\, d\overline{x}.$$

Here σ_{ij} and ϵ_{ij}, $j = 1, 2, 3$ denote the stress and strain tensors, respectively. For a small displacement theory we assume that

$$\epsilon_{jk}(\overline{x}) = \frac{1}{2}\left(\frac{\partial U_j(\overline{x})}{\partial x_k} + \frac{\partial U_k(\overline{x})}{\partial x_j}\right), \quad \forall \overline{x} \in Q.$$

It then follows from the kinematic relations that the strains within the ith layer are

$$\epsilon_{11} = \frac{\partial v^i}{\partial x_1} + (x_3 - \hat{z}_i)\frac{\partial \psi^i}{\partial x_1}, \quad \epsilon_{13} = \epsilon_{31} = \frac{1}{2}(\psi^i + w_x) = \frac{1}{2}\varphi^i, \tag{4}$$

and the other strains vanish. If the beam layers are assumed homogeneous and transversely isotropic (although material properties may vary from layer to layer), then by making the further assumption that σ_{33} is negligible we can follow Mindelin's approach to obtain the following constitutive equations within each layer:

$$\sigma_{11} = \frac{E_i}{1-\nu_i^2}\epsilon_{11}, \quad \sigma_{13} = \sigma_{31} = 2G_i\epsilon_{13}. \tag{5}$$

Here E_i denotes the in-plane Young's modulus, ν_i represents the in-plane Poisson's ratio and G_i represents the transverse shear modulus (with shear correction included [9]), all for the ith layer. In our model $G_1 = G_3 = 0$, and we write simply $G_2 = G$. From (4) and (5) the strain energy of the ith layer can be written as

$$\mathcal{P}_i = r\frac{1}{2}\int_0^l D_i h_i^3 \left(\frac{\partial \psi^i}{\partial x_1}\right)^2 + 12 D_i h_i \left(\frac{\partial v^i}{\partial x_1}\right)^2 + h_i G_i (\varphi^i)^2\, dx_1,$$

where $D_i = E_i/12(1-\nu_i^2)$. Thus $D_i h_i^3$ is the modulus of flexural rigidity for the ith layer, and Gh_2 is the modulus of elasticity in shear for the middle layer. Since the longitudinal variable x_1 is the only spatial variable remaining in the kinetic and potential energy (the other spatial directions have been integrated out), let us simplify notation a bit and use x for x_1 from now on.

It is often convenient to express the kinetic and potential energies in terms of certain bilinear forms. To this end define the matrices

$$\begin{aligned} \mathbf{h} &= \mathrm{diag}\,(h_1, h_2, h_3), & \mathbf{p} &= \mathrm{diag}\,(\rho_1, \rho_2, \rho_3), \\ \mathbf{D} &= \mathrm{diag}\,(D_1, D_2, D_3), & \mathbf{G} &= \mathrm{diag}\,(0, G, 0). \end{aligned}$$

Now define the bilinear forms

$$a([\psi, v, \varphi]; [\hat{\psi}, \hat{v}, \hat{\varphi}]) \;=\; \int_0^l \mathbf{D}\mathbf{h}^3 \frac{d\psi}{dx}\cdot\frac{d\hat{\psi}}{dx} + 12\mathbf{D}\mathbf{h}\frac{dv}{dx}\cdot\frac{d\hat{v}}{dx} + \mathbf{G}\mathbf{h}\varphi\cdot\hat{\varphi}\, dx \tag{6}$$

$$c([\psi, v, w]; [\hat{\psi}, \hat{v}, \hat{w}]) \;=\; \int_0^l m w\hat{w} + \frac{1}{12}\mathbf{p}\mathbf{h}^3\psi\cdot\hat{\psi} + \mathbf{p}\mathbf{h}v\cdot\hat{v}\, dx, \tag{7}$$

where $f \cdot g$ means the usual dot product when $f, g \in \mathbb{R}^3$. The potential and kinetic energies can now be written as

$$\mathcal{K} = \frac{r}{2}c([\dot{\psi}, \dot{v}, \dot{w}]; [\dot{\psi}, \dot{v}, \dot{w}]), \qquad \mathcal{P} = \frac{r}{2}a([\psi, v, \varphi]; [\psi, v, \varphi]).$$

2.3. **Equations of Motion.** For simplicity we assume there are no external forces acting on the beam. Thus the Langrangian is given by

$$\mathcal{L} = \int_0^T [\mathcal{K}(t) - \mathcal{P}(t)] \, dt.$$

To obtain the equations of motion for a particular set of boundary conditions, one applies Hamilton's principle – the variation of the Lagrangian is set to zero, where the variation is taken with respect to all kinematically admissible motions. This leads to the equations of motion in weak form. To obtain a boundary value problem one must select an appropriate set of state variables. For the general case of n-layers with each layer modeled as a Reissner-Mindlin plate, these calculations can be found in [3]. The resulting boundary value problem is a system of $2n+3$ partial differential equations. For the generalization to n plates of the 3-layer beam considered here, we refer to [2]. For the particular 3-layer beam under consideration we choose w, v^1, v^3 as state variables. All other variables can be written in terms of these, and in particular

$$\begin{aligned}
\psi^2 &= -\tfrac{1}{h_2} v^1 + \tfrac{1}{h_2} v^3 + \tfrac{h_1+h_3}{2h_2} w_x \\
\varphi^2 &= -\tfrac{1}{h_2} v^1 + \tfrac{1}{h_2} v^3 + \tfrac{1}{h_2} H w_x, \\
v^2 &= \tfrac{1}{2} v^1 + \tfrac{1}{2} v^3 + \tfrac{h_3-h_1}{4} w_x,
\end{aligned} \tag{8}$$

where we defined $H = h_2 + (h_1+h_3)/2$. Then we obtain the following equations of motion:

$$\begin{aligned}
m\ddot{w} &- \left\{ -\frac{\rho_2 h_2}{12}(2h_1 - h_3)\ddot{v}^1 + D_2 h_2 (2h_1 - h_3) v^1_{xx} \right\}_x \\
&- \left\{ \frac{\rho_2 h_2}{12}(2h_3 - h_1)\ddot{v}^3 - D_2 h_2 (2h_3 - h_1) v^3_{xx} \right\}_x \\
&- \left\{ [\frac{\rho_1 h_1^3 + \rho_3 h_3^3}{12} + \frac{\rho_2 h_2}{12}(h_1^2 + h_3^2 - h_1 h_3)]\ddot{w} \right. \\
&\qquad \left. - [D_1 h_1^3 + D_3 h_3^3 + D_2 h_2 (h_1^2 + h_3^2 - h_1 h_3)] w_{xx} \right\}_{xx} \\
&+ \frac{HG}{h_2}(v^1 - v^3 - H w_x)_x = 0,
\end{aligned} \tag{9}$$

$$\begin{aligned}
(\rho_1 h_1 + \frac{1}{3}\rho_2 h_2)\ddot{v}^1 &- (12 D_1 h_1 + 4 D_2 h_2) v^1_{xx} + \frac{1}{6}(\rho_2 h_2)\ddot{v}^3 - (2 D_2 h_2) v^3_{xx} \\
&- \frac{\rho_2 h_2}{12}(2h_1 - h_3)\ddot{w}_x + D_2 h_2 (2h_1 - h_3) w_{xxx} + \frac{HG}{h_2}(v^1 - v^3 - H w_x) = 0,
\end{aligned} \tag{10}$$

$$\begin{aligned}
\frac{1}{6}(\rho_2 h_2)\ddot{v}^1 &- (2 D_2 h_2) v^1_{xx} + (\rho_3 h_3 + \frac{1}{3}\rho_2 h_2)\ddot{v}^3 - (12 D_3 h_3 + 4 D_2 h_2) v^3_{xx} \\
&+ \frac{\rho_2 h_2}{12}(2h_3 - h_1)\ddot{w}_x - D_2 h_2 (2h_3 - h_1) w_{xxx} - \frac{HG}{h_2}(v^1 - v^3 - H w_x) = 0,
\end{aligned} \tag{11}$$

plus appropriate boundary conditions. For a cantilevered beam, the boundary conditions are

$$
\begin{aligned}
& w(t,0) = w_x(t,0) = v^1(t,0) = v^3(t,0) = 0, \\
& \Big(-\frac{\rho_2 h_2}{12}(2h_1 - h_3)\ddot{v}^1 + D_2 h_2 (2h_1 - h_3) v^1_{xx} + \frac{\rho_2 h_2}{12}(2h_3 - h_1)\ddot{v}^3 \\
& \qquad - D_2 h_2 (2h_3 - h_1) v^3_{xx} + [\frac{\rho_1 h_1^3 + \rho_3 h_3^3}{12} + \frac{\rho_2 h_2}{12}(h_1^2 + h_3^2 - h_1 h_3)]\ddot{w} \\
& \qquad - [D_1 h_1^3 + D_3 h_3^3 + D_2 h_2 (h_1^2 + h_3^2 - h_1 h_3)] w_{xx} \\
& \qquad - \frac{HG}{h_2}(v^1 - v^3 - H w_x) \Big) \Big|_{x=l} = 0, \\
& \big(-D_2 h_2 (2h_1 - h_3) v^1_x + D_2 h_2 (2h_3 - h_1) v^3_x \\
& \qquad + [D_1 h_1^3 + D_3 h_3^3 + D_2 h_2 (h_1^2 + h_3^2 - h_1 h_3)] w_{xx} \big) \big|_{x=l} = 0, \\
& \big((12 D_1 h_1 + 4 D_2 h_2) v^1_x + 2(D_2 h_2) v^3_x - D_2 h_2 (2h_1 - h_3) w_{xx} \big) \big|_{x=l} = 0, \\
& \big(2(D_2 h_2) v^1_x + (12 D_3 h_3 + 4 D_2 h_2) v^3_x + D_2 h_2 (2h_3 - h_1) w_{xx} \big) \big|_{x=l} = 0.
\end{aligned}
$$

Boundary conditions for other beam configurations can be similarly derived. The equations of motion (9)-(11) can also be found in [10], which was one of the first papers to include the effects of transverse, longitudinal, and rotational inertia for unsymmetric three-layered structures. In several earlier studies, most notably [11], [1], [8] and [13], simpler equations of motion were derived by either neglecting certain inertia terms or by assuming some symmetry of the outer layers. A comparison of the various models can be found in [7] and [12]. We shall proceed in this fashion and neglect the effects of longitudinal and rotatory inertia. We will then study properties of the solution semigroup. When damping (due to shear in the middle layer) is included in the model, we use an eigenvalue analysis to derive an expression for the optimal damping based on the geometric and material parameters of the structure.

To proceed, when the effects of longitudinal and rotational inertia are neglected, the kinetic energy becomes

$$\mathcal{K} = \frac{r}{2}\int_0^l m(\dot{w})^2\,dx.$$

Before writing the simplified equations of motion, we shall find it convenient to introduce the new geometric variable $s = h_2\varphi^2$ and new material parameter $\gamma = G/h_2$. We see from (8) that

$$s = -(v^1 - v^3 - H w_x).$$

Now if we use the simplified kinetic energy and apply Hamilton's principle, we get the following equations of motion:

$$
\begin{aligned}
& m\ddot{w} - D_2 h_2 (2h_1 - h_3) v^1_{xxx} + D_2 h_2 (2h_3 - h_1) v^3_{xxx} \\
& \qquad + [D_1 h_1^3 + D_3 h_3^3 + D_2 h_2 (h_1^2 + h_3^2 - h_1 h_3)] w_{xxxx} - H\gamma s = 0, && (12) \\
& (4 D_2 h_2 + 12 D_1 h_1) v^1_{xx} + 2 D_2 h_2 v^3_{xx} - D_2 h_2 (2h_1 - h_3) w_{xxx} + \gamma s = 0, && (13) \\
& D_2 h_2 v^1_{xx} + (4 D_2 h_2 + 12 D_3 h_3) v^3_{xx} + D_2 h_2 (2h_3 - h_1) w_{xxx} - \gamma s = 0. && (14)
\end{aligned}
$$

From this dynamic bending equation and two static stretching equations, it is possible to eliminate the stretching variables v^1, v^3 and obtain the following pair of equations:

$$
\begin{aligned}
m\ddot{w} + A w_{xxxx} - B\gamma s_x &= 0, && (15) \\
C\gamma s - s_{xx} + B w_{xxx} &= 0, && (16)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= D_1h_1^3 + D_3h_3^3 \\
&\quad + \frac{D_2h_2[3D_2h_2(D_1h_1^3 + D_3h_3^3) + 12D_1D_3h_1h_3(h_1^2 + h_3^2 - h_1h_3)]}{D_2h_2(4D_1h_1 + 4D_3h_3 + D_2h_2) + 12D_1D_3h_1h_3}, \\
B &= \frac{D_2h_2(3D_1h_1^2 + D_2h_2^2 + 3D_3h_3^2 + 4D_1h_1h_2 + 4D_3h_3h_2) + 12D_1D_3Hh_1h_3}{D_2h_2(4D_1h_1 + 4D_3h_3 + D_2h_2) + 12D_1D_3h_1h_3}, \\
C &= \frac{D_1h_1 + D_2h_2 + D_3h_3}{D_2h_2(4D_1h_1 + 4D_3h_3 + D_2h_2) + 12D_1D_3h_1h_3}.
\end{aligned}
$$

For a cantilevered beam the boundary conditions for (15)-(16) are

$$
\begin{array}{lll}
w(t,0) = 0, & w_x(t,0) = 0, & s(t,0) = 0, \\
w_{xx}(t,l) = 0, & s_x(t,l) = 0, & Aw_{xxx}(t,l) - B\gamma s(t,l) = 0.
\end{array}
$$

Note that if we solve for s_x and s_{xxx} in (15) and substitute into (16) (differentiated once), we get the following sixth order equation:

$$(m\ddot{w} + Aw_{xxxx})_{xx} - C\gamma(m\ddot{w} + [A + \frac{B^2}{C}]w_{xxxx}) = 0. \tag{17}$$

This may be considered a slight generalization of the familiar Mead-Markus model [8], since we have allowed for longitudinal stress in the core (ie., we have $D_2 \neq 0$). In fact, if we set $D_2 = 0$, then

$$A = D_1h_1^3 + D_3h_3^3, \quad B = H, \quad C = \frac{D_1h_1 + D_3h_3}{12D_1D_3h_1h_3},$$

and (17) is exactly the Mead-Markus model.

3. **Analysis of Equations of Motion.**

3.1. **The undamped model.** Observe that for fixed t the static equation (16) can be written as

$$(\alpha I - T)s = -Bw''', \tag{18}$$

where $\alpha = \gamma C > 0$ and T is the second order differential operator defined on the domain $\operatorname{dom} T = \{\varphi \in H^2(0,l) : \varphi(0) = \varphi'(l) = 0\}$ by $T\varphi = \varphi''$. T is a densely defined, self-adjoint, negative definite, unbounded linear operator on $L^2(0,l)$, so the resolvent $(\alpha I - T)^{-1}$ exists and is a bounded operator defined on all of $L^2(0,l)$. Now define $J = -I + \alpha(\alpha I - T)^{-1}$, which is self-adjoint and bounded on $L^2(0,l)$, and also satisfies $J = (\alpha I - T)^{-1}T$ on $\operatorname{dom} T$. In other words, J is the bounded extension of $(\alpha I - T)^{-1}T$ from $\operatorname{dom} T$ to all of $L^2(0,l)$. Furthermore, J is non-positive on $L^2(0,l)$, that is,

$$\int_0^l (Jz)\overline{z}\,dx \leq 0 \text{ for all } z \in L^2(0,l). \tag{19}$$

To see this, let $z \in L^2(0,l)$ and set $y = (\alpha I - T)^{-1}z$, so that $y \in \operatorname{dom} T$ and $\alpha y - y'' = z$. Then

$$
\begin{aligned}
\int_0^l (Jz)\overline{z}\,dx &= \int_0^l (-z + \alpha(\alpha I - T)^{-1}z)\overline{z}\,dx \\
&= \int_0^l (-(\alpha y - y'') + \alpha y)(\alpha\overline{y} - \overline{y''})\,dx \\
&= \int_0^l y''(\alpha\overline{y} - \overline{y''})\,dx \\
&= -\int_0^l \alpha|y'|^2 + |y''|^2\,dx \leq 0,
\end{aligned}
$$

so (19) is true. Also, since J is bounded on $L^2(0,l)$, there exists a constant $K > 0$ such that

$$\int_0^l -(Jz)\overline{z}\,dx \le K \int_0^l |z|^2\,dx \tag{20}$$

for all $z \in L^2(0,l)$.

When we solve (18) for s and substitute into (15), we obtain the equation

$$m\ddot{w} + Aw_{xxxx} + B^2\gamma[(\alpha I - T)^{-1}w_{xxx}]_x = 0, \tag{21}$$

with (cantilevered) boundary conditions

$$\begin{aligned} &w(t,0) = w_x(t,0) = 0,\\ &w_{xx}(t,l) = Aw_{xxx}(t,l) + B^2\gamma(\alpha I - T)^{-1}w_{xxx}(t,l) = 0. \end{aligned} \tag{22}$$

The total energy of a solution is seen to be

$$E(t) = \frac{1}{2}\int_0^l m(\dot{w})^2 + A(w_{xx})^2 - (B^2\gamma(\alpha I - T)^{-1}w_{xxx})w_x\,dx. \tag{23}$$

Now (21) can be reformulated as an abstract Cauchy problem in a natural way. In particular, define the state space

$$X = H_L^2(0,l) \times L^2(0,l),$$

where $H_L^2(0,l) = \{\varphi \in H^2(0,l) :\ \varphi(0) = \varphi'(0) = 0\}$. Motivated by (23) we define the energy norm $\|\cdot\|_e$ on X by

$$\|(u,v)\|_e^2 = \int_0^l m|v|^2 + A|u''|^2 - B^2\gamma(Ju')\overline{u'}\,dx,$$

which has a compatible inner product

$$\langle (u,v),(f,g)\rangle_e = \int_0^1 mv\overline{g} + Au''\overline{f''} - B^2\gamma(Ju')\overline{f'}\,dx \tag{24}$$

Next define the operator $\mathcal{A} : \operatorname{dom}\mathcal{A} \subset X \to X$ on the domain

$$\begin{aligned} \operatorname{dom}\mathcal{A} = \{(u,v) \in X :\ &v \in H_L^2(0,l),\ u' \in \operatorname{dom} T,\\ &[Au''' + B^2\gamma(Ju')] \in H_R^1(0,l)\} \end{aligned} \tag{25}$$

by

$$\mathcal{A}(u,v) = (v, -\frac{1}{m}[Au''' + B^2\gamma(Ju')]'). \tag{26}$$

Here we have defined $H_R^1(0,l) = \{\varphi \in H^1(0,l) :\ \varphi(l) = 0\}$. Now set $x(t) = (w(t,x),\dot{w}(t,x))$, and (21)-(22) (and hence (15)-(16)) can be reformulated as the Cauchy problem

$$\frac{d}{dt}x(t) = \mathcal{A}x(t) \quad \text{on } X, \tag{27}$$

with an appropriate initial condition $x(0) = x_0$. We show that (27) is well-posed by verifying that $\mathcal{A}$ generates a C_0-semigroup on X.

Theorem 1. *The operator $\mathcal{A}$ defined by (25)-(26) is the infinitesimal generator of a strongly continuous semigroup of contractions on X.*

Proof: We shall first show that $\mathcal{A}$ is dissipative on X with the inner product (24), and then that the range condition $\operatorname{ran}(I-\mathcal{A}) = X$ is satisfied. The result then follows from the Lumer-Phillips theorem. To proceed, let $(u,v) \in \operatorname{dom} A$. Then

$$\begin{aligned}
\operatorname{Re}\langle \mathcal{A}(u,v),(u,v)\rangle_e &= \operatorname{Re}\int_0^l Av''\overline{u''} - [Au''' + B^2\gamma Ju']'\overline{v} - B^2\gamma Jv'\overline{u'}\,dx \\
&= \operatorname{Re}\int_0^l Av''\overline{u''} + [Au''' + B^2\gamma(Ju')]\overline{v'} - B^2\gamma Jv'\overline{u'}\,dx \\
&= \operatorname{Re}\int_0^l Av''\overline{u''} - Au''\overline{v''} + B^2\gamma Ju'\overline{v'} - B^2\gamma Jv'\overline{u'}\,dx \\
&= 0,
\end{aligned}$$

where we used that J is self-adjoint. In order to prove the range condition, let $(f,g) \in X$. We must show that there exists $(u,v) \in \operatorname{dom}\mathcal{A}$ for which

$$u - v = f, \tag{28}$$

$$v + \frac{1}{m}[Au''' + B^2\gamma Ju']' = g. \tag{29}$$

Solve (28) for v and substitute in (29) to get

$$mu + [Au''' + B^2\gamma Ju']' = mf + mg. \tag{30}$$

We are finished once we show that (30) has a solution, but this is equivalent to showing that the range of $mI - S$ is all of $L^2(0,l)$, where S is defined on the domain

$$\operatorname{dom} S = \{u \in H^3(0,l) \cap H_L^2(0,l) : \; u''(l) = 0, \; Au''' + B^2\gamma Ju' \in H_R^1(0,l)\}$$

by $Su = -[Au''' + B^2\gamma Ju']'$. To see that this is true, we consider the sesquilinear form σ defined on $H_L^2(0,l)$ by

$$\sigma(u,w) = \int_0^l Au''\overline{w''} - B^2\gamma Ju'\overline{w'}\,dx.$$

When $H_L^2(0,l)$ is endowed with the norm

$$\| u \|_{H_L^2} = \int_0^l A|u''|^2\,dx,$$

it is easy to see that $H_L^2(0,l)$ is densely and compactly embedded in $L^2(0,l)$, and that σ is coercive and bounded on $H^2(0,l)$. By the Lax-Milgram theorem σ defines an operator from $H_L^2(0,l)$ onto its dual, and the restriction to $L^2(0,l)$ (assuming a Gelfand triple framework) is precisely the operator S. Thus $(mI - S)$ is onto, and the result follows.

3.2. **The damped model.** We shall find it more convenient to study (15)-(16) in another form. After differentiating (16) and substituting into (15), we get

$$m\ddot{w} + (A + \frac{B^2}{C})w_{xxxx} - \frac{B}{C}s_{xxx} = 0, \tag{31}$$

$$\gamma s - \frac{1}{C}s_{xx} + \frac{B}{C}w_{xxx} = 0. \tag{32}$$

For a cantilevered beam we have the boundary conditions

$$\begin{aligned}
&w(t,0) = w_x(t,0) = s(t,0) = 0, \\
&w_{xx}(t,l) = s_x(t,l) = 0, \; (A + \tfrac{B^2}{C})w_{xxx}(t,l) - \tfrac{B}{C}s_{xx}(t,l) = 0.
\end{aligned} \tag{33}$$

We are interested in the effect of damping due to shear in the middle layer. In particular we will assume a constitutive law in the core layer for which the shear restoring force is a linear combination of the shear strain and the shear strain rate. Thus we replace the constitutive equation $\sigma_{13} = 2G\epsilon_{13}$ with

$$\sigma_{13} = 2(G\epsilon_{13} + G_d\dot{\epsilon}_{13}).$$

If we apply the viscoelastic correspondence principle we find that the equations of motion are modified to

$$\begin{aligned} m\ddot{w} + (A + \frac{B^2}{C})w_{xxxx} - \frac{B}{C}s_{xxx} &= 0, && (34)\\ \beta\dot{s} + \gamma s - \frac{1}{C}s_{xx} + \frac{B}{C}w_{xxx} &= 0, && (35)\end{aligned}$$

with the same boundary conditions (33). The damping parameter β is related to G_d in the same way that γ is related to G. The total energy of a solution is now given by

$$E(t) = \frac{1}{2}\int_0^l m(\dot{w})^2 + A(w_{xx})^2 + \gamma(s)^2 + \frac{1}{C}(Bw_{xx} - s_x)^2\,dx. \tag{36}$$

Now, the natural state space framework is quite similar in mathematical structure to that for the damped 'adhesive-layer' beam considered in [4] and [5]. Without providing details, we observe that an argument similar to the one in [4] can be used to show that the solution semigroup for (34)-(35) is in fact analytic.

3.3. **Optimal damping.** In this section we derive a formula for the optimal choice of the damping parameter β in terms of the material parameters of the structure. We do this by studying the effect of β on the location of the system eigenvalues in the complex plane. For purposes of this eigenvalue analysis, it will be convenient to consider a simply supported beam, in which case the boundary conditions (33) are replaced by

$$\begin{aligned} &w(t,0) = w(t,l) = w_{xx}(t,0) = w_{xx}(t,l) = 0,\\ &s_x(t,0) = s_x(t,l) = 0.\end{aligned} \tag{37}$$

The eigenfunctions associated with (31), (32), (37) are purely sinusoidal, and an equation for the eigenvalues s_k is obtained by assuming solutions of the form

$$w = e^{s_k t}\sin(\alpha_k x), \quad s = Ke^{s_k t}\cos(\alpha_k x),$$

where $\alpha_k = k\pi/l$. If we substitute into (31), (32) and then eliminate the constant K, we obtain

$$\beta m s_k^3 + (\frac{\alpha_k^2}{C} + \gamma)m s_k^2 + \beta\frac{AC+B^2}{C}\alpha_k^4 s_k + \frac{AC+B^2}{C}\alpha_k^4(\gamma + \frac{A}{AC+B^2}\alpha_k^2) = 0 \tag{38}$$

(compare with (35) in [5]).

Remark Let us observe that when $\beta = 0$ (no damping), there are two roots $s_k^{\pm} = \pm i\sigma_k$ of (38) for each $k = 1, 2, \ldots$, where

$$\sigma_k = \sqrt{\frac{\gamma(AC+B^2) + A\alpha_k^2}{(\alpha_k^2 + C\gamma)m}}\,\alpha_k^2. \tag{39}$$

The flexural wave speeds are defined by $v_k = \sigma_k/\alpha_k$. One sees that $v_k \to \sqrt{(AC+B^2)/(Cm)}\alpha_k$ if $\gamma \to \infty$, the same as for an Euler-Bernoulli beam satisfying $m\ddot{w} + (A + B^2/C)w_{xxxx} = 0$. If $\gamma \to 0$, the wave speeds satisfy $v_k \to \sqrt{A/m}\alpha_k$.

Now let us consider the case $\beta \neq 0$ (damping included). In order to choose an 'optimal' value of the damping parameter β, we must understand how the roots of (38) behave for all $\beta \in (0, \infty)$.

Let us introduce the change of variables

$$x = \sqrt{\frac{m}{A}}\frac{1}{\alpha_k^2}s_k, \tag{40}$$

$$\overline{\beta} = \sqrt{\frac{m}{A}}\frac{1}{C\beta}, \qquad \delta = \frac{AC+B^2}{AC} = 1 + \frac{B^2}{AC} > 1.$$

It is straightforward to check that s_k is a root of (38) if and only if x is a root of

$$x^3 + \overline{\beta}x^2 + \delta x + \overline{\beta} + \frac{\overline{\beta}C\gamma}{\alpha_k^2}(x^2+\delta) = 0. \tag{41}$$

We observe that as β takes all values from 0 to ∞, so does $\overline{\beta}$, and vice-versa. Thus it will be sufficient to consider the roots of (41) for values of $\overline{\beta} \in (0,\infty)$. We further observe that for high frequencies (i.e. asymptotically as $\alpha_k \to \infty$), Rouche's theorem implies that the roots of (41) for $\gamma \neq 0$ will be close to the roots of (41) with $\gamma = 0$. Thus, let us first consider the case $\gamma = 0$ (the shear stiffness of the middle layer is neglected). Then we are interested in the roots of

$$f(x) = x^3 + \overline{\beta}x^2 + \delta x + \overline{\beta} = 0 \tag{42}$$

for $\overline{\beta} \in (0,\infty)$, where $\delta > 1$. Since $f(0) > 0$ and $f'(x) > 0$ for $x > 0$, all real roots of (42) are negative. Also $f(-\overline{\beta}) < 0$ and $f(-\overline{\beta}/\delta) > 0$, so there is at least one real root on $(-\overline{\beta}, -\overline{\beta}/\delta)$, which we denote by $a = a(\overline{\beta},\delta)$. If there is a complex conjugate pair of roots, say $b \pm ci$, then $f(x) = (x-a)(x-(b+ci))(x-(b-ci))$ implies that $a + 2b = -\overline{\beta}$, hence $b < 0$. Thus all roots of (42) lie in the left half complex plane.

Each root of (38) with $\gamma = 0$ is a real multiple (by (40)) of one of the three roots of (42). Thus there are three branches of eigenvalues, and the argument of the branches is determined by the argument of the roots of (42). This is illustrated in Figure 1.

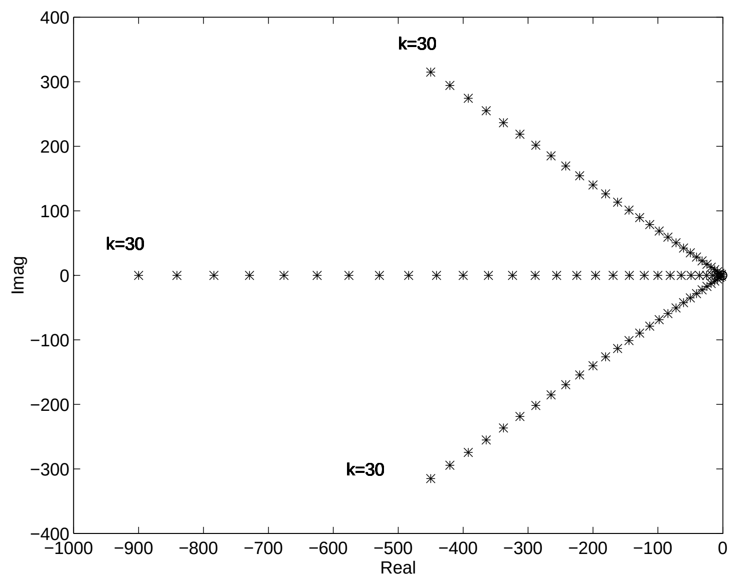

FIGURE 1. Typical location of eigenvalues of (31)-(32)

We are interested in how to choose an optimal $\overline{\beta} \in (0,\infty)$ for a given fixed δ (ie. for given fixed material parameters). It can be shown that if $\delta > 9$, there is always a value of $\overline{\beta}$ for which the roots of (42) are all real (the system is completely overdamped), and for $1 < \delta \leq 9$ there is a complex conjugate pair of roots of (42) for all values of $\overline{\beta} \in (0, \infty)$. Thus let us restrict our attention to $1 < \delta \leq 9$, and let $v(\overline{\beta}) = b + ci$ be the complex root of (42) with positive imaginary part. Figure 2 shows a typical plot of $v(\overline{\beta})$ as $\overline{\beta}$ goes from 0 to ∞. For a fixed $\delta \in (1, 9]$, the

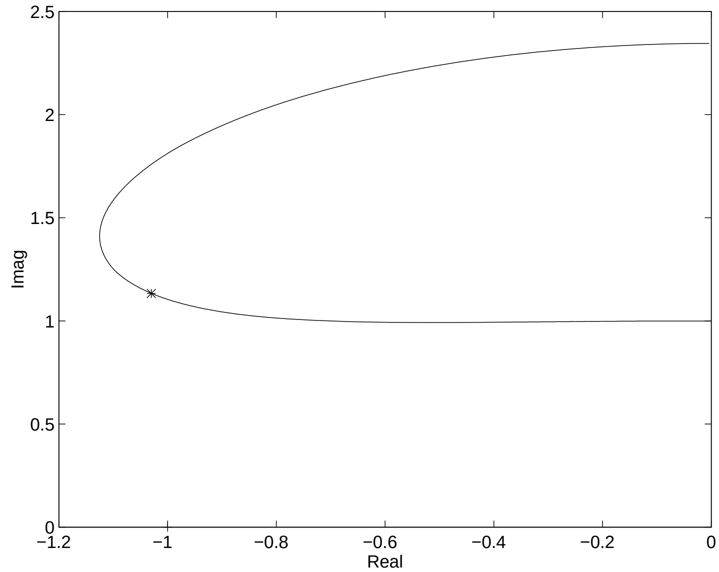

FIGURE 2. Typical values of $v(\overline{\beta})$ as $\overline{\beta}$ goes from 0 to ∞, with optimal value β^* indicated by $*$.

optimal value of $\overline{\beta}$ means the value which maximizes the argument of $v(\overline{\beta})$ over all $\overline{\beta} \in (0, \infty)$. Since for $1 < \delta \leq 9$ the roots of (42) stay separated for all $\overline{\beta} \in (0, \infty)$, it follows from results in [6] that $v(\overline{\beta})$ is an analytic function. Thus $\arg(v(\overline{\beta}))$ is smooth, and we further observe that $\arg(v(\overline{\beta})) \to \pi/2$ as $\overline{\beta} \to 0$ and as $\overline{\beta} \to \infty$. This implies that $\arg(v(\overline{\beta}))$ has an absolute maximum value.

Theorem 2. *Let $\delta \in (1, 9]$. Then $\beta^* = \delta^{3/4}$ is the location of the absolute maximum of* $\arg(v(\overline{\beta}))$.

Proof: Set $\beta^* = \delta^{3/4}$. We will show that for any $\beta_1 > \beta^*$ (resp. $\beta_1 < \beta^*$), there exists $\beta_2 < \beta^*$ (resp. $\beta_2 > \beta^*$) such that $\arg(v(\beta_1)) = \arg(v(\beta_2))$ and $v(\beta_1) \neq v(\beta_2)$. The result then follows from geometric consideration of the curve $\arg(v(\overline{\beta}))$. To proceed, suppose that $\beta_1 > \beta^*$ (the proof for the other case is similar). For this value β_1, denote the real root of (42) by $a_1 = a_1(\beta_1, \delta)$ and denote the complex roots by $b_1 \pm c_1 i$. In particular,

$$(b_1 - c_1 i)^3 + \beta_1 (b_1 - c_1 i)^2 + \delta (b_1 - c_1 i) + \beta_1 = 0. \tag{43}$$

We claim that

$$a_1 \neq -\frac{\beta_1}{\delta^{1/2}}. \tag{44}$$

If equality holds in (44), then $f(a_1) = 0$ implies

$$-\frac{\beta_1^3}{\delta^3/2} + \frac{\beta_1^3}{\delta} - \beta_1 \delta^{1/2} + \beta_1 = 0,$$

so that

$$\frac{\beta_1^3}{\delta^{3/2}}(\delta^{1/2} - 1) - \beta_1(\delta^{1/2} - 1) = 0.$$

Hence $\beta_1 = \delta^{3/4} = \beta^*$, contradicting that $\beta_1 > \beta^*$. Now since $f(x) = (x - a_1)(x - (b_1 + c_1 i))(x - (b_1 - c_1 i))$, we see that $\beta_1 = -a_1(b_1^2 + c_1^2)$. Thus we have

$$\frac{\delta^{1/2}}{b_1^2 + c_1^2} = -\frac{a_1 \delta^{1/2}}{\beta_1} \neq 1. \tag{45}$$

Next set $\beta_2 = \delta^{3/2}/\beta_1$, and note that $\beta_2 = (\beta^*/\beta_1)\beta^* < \beta^*$. Then $\delta^{1/2}/(b_1 - c_1 i)$ is a root of (42) for $\overline{\beta} = \beta_2$, since

$$\begin{aligned} f(\frac{\delta^{1/2}}{(b_1 - c_1 i)}) &= \frac{\delta^{3/2}}{(b_1 - c_1 i)^3} + \frac{\delta^{3/2}}{\beta_1}\frac{\delta}{(b_1 - c_1 i)^2} + \delta \frac{\delta^{1/2}}{b_1 - c_1 i} + \frac{\delta^{3/2}}{\beta_1} \\ &= \left[\frac{\delta^{3/2}}{\beta_1 (b_1 - c_1 i)^3}\right]\left[(b_1 - c_1 i)^3 + \beta_1 (b_1 - c_1 i)^2 + \delta(b_1 - c_1 i) + \beta_1\right] \\ &= 0 \end{aligned}$$

by (43). But $v(\beta_1) = b_1 + c_1 i$ and

$$\frac{\delta^{1/2}}{b_1 - c_1 i} = \frac{\delta^{1/2}}{b_1^2 + c_1^2} v(\beta_1),$$

so $v(\beta_2) = \frac{\delta^{1/2}}{b_1^2 + c_1^2} v(\beta_1)$. Thus $v(\beta_1)$ and $v(\beta_2)$ have the same argument, but by (45) they have different modulus, and the result follows.

The previous result shows that $\beta^* = \delta^{3/4}$ is the optimal choice of $\overline{\beta}$, and the optimal angle is $\theta^* = \arg(v(\beta^*))$. One can check that for this choice of $\overline{\beta}$, the real root is given by $a(\beta^*) = -\delta^{1/4}$, and the complex roots are given by

$$\frac{\delta^{1/4}(1 - \delta^{1/2})}{2} \pm \frac{\delta^{1/4}\sqrt{(3 - \delta^{1/2})(1 + \delta^{1/2})}}{2} i.$$

Thus the optimal angle is

$$\theta^* = \pi - \tan^{-1}\left(\frac{\sqrt{(3 - \delta^{1/2})(1 + \delta^{1/2})}}{\delta^{1/2} - 1}\right).$$

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