Analysis of a Plate with a Localized Piezoelectric Patch

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Abstract

We first describe an approach for the modeling of piezoelectric beams and plates that leads to a well-posed system of partial differential equations that retain the coupling between the mechanical system and a potential equation for the electrical components. We then describe how the same approach can be used to analyze a plate with a localized piezoelectric patch. In particular, since the higher order electrical terms have been retained, it is possible to deduce the proper sets of boundary conditions that should apply to an arbitrarily shaped patch.

1 Introduction

Piezoelectric theory dates back to the 1880's when the Curie brothers first detected the so called "direct" piezoelectric effect and Lippman who soon after predicted the "converse" peizoelectric effect. Since then various mathematical models for piezoelectric materials have been proposed. Early piezoelastic beam and plate models include the work of Mindlin and Tiersten in the 60's ([5], [4]). As engineering applications have emerged in the 1980's and 90's, more elaborite and specialized models have been produced. The books by Tzou [6] and Rogacheva [3] describe several possible approaches in the modeling of both general piezoelectric materials and piezoelectric patches for the purpose of controlling vibration in plates and shells.

A common assumption used in modeling piezoelectric patches on a plate (or shell) is to eliminate the peizoelectric coupling by solving for the electrical components of the piezoelectric equations under a stress-free mechanical assumption. One can then back-substitute the result in the mechanical equations. This produces an uncoupled system, in which an electrical input directly influences the mechanical components. This approximation turns out to be reasonably valid except in a thin boundary layer region near the edge of the patch. However, precise knowledge of the conditions near the boundary of the patch are necessary to form the proper boundary conditions and also to allow the problem to be posed in the natural energy space.

The goal of this research is to (i) analyze the differences that result in retaining the higher-order electroelastic coupling terms in the patch, (ii) derive through the variational approach the appropriate boundary conditions for a patch of arbitrary shape (with some reasonable limitations) in a plate, (iii) provide a foundation from which it is possible to analyze the well-posedness of the plate-patch system under various possible electrical boundary conditions.

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2 Patch Modeling

The patch is viewed as a thin piezoelastic plate of thickness h_1 in which the upper and lower surfaces are covered with a foil that serves as the electrodes. We assume that the input voltage $V(t)$ produces a uniform charge distribution of $\pm r(t)/2$ per unit area on the upper and lower electrodes. One begins with the piezoelectric constitutive equations:

$$\begin{aligned} \{T_{ij}\} &= [c]\{S_{ij}\} - [e]^T\{E_j\} \\ \{D_j\} &= [e]\{S_{ij}\} + [\epsilon]\{E_j\}. \end{aligned} \tag{2.1}$$

In (2.1), T represents the (second order) stress tensor, S the (second order) strain tensor, E the electric field, D the electric displacement vector. The tensors $[c]$, $[e]$ and $[\epsilon]$ depend upon the material properties of the patch. Since the tensors S and T are symmetric, they can be viewed as vectors with six components S_{11}, S_{22}, S_{33}, S_{12}, S_{13}, S_{23}, and likewise for T. Various simplifying assumptions are typically made at this point to obtain a one-dimensional beam theory or two-dimensional plate theory. To introduce ideas we begin with the beam theory.

2.1 Piezoelectric Beams

In the case of a beam we ignore the forces acting in the y direction. (We assume that z is the transverse variable and x the longitudial variable). Thus the corresponding elements of the S, T, E and D vectors may be ignored. We can then write

$$\begin{aligned} \{T\} &= (T_{11}, T_{33}, T_{13})^T, \quad & S &= (S_{11}, S_{33}, S_{13})^T \\ \{E\} &= (E_1, E_3)^T & \{D\} &= (D_1, D_3)^T. \end{aligned} \tag{2.2}$$

The tensors c, ϵ, e may thus be represented as matrices.

Under typical assumptions (transversely isotropy, with polorization in the z direction, among others) these matrices have the form

$$\begin{pmatrix} [c] & [e]^T \\ [e] & [\epsilon] \end{pmatrix} = \begin{pmatrix} c_{11} & c_{13} & 0 & 0 & e_{31} \\ c_{13} & c_{33} & 0 & 0 & e_{33} \\ 0 & 0 & c_{55} & e_{15} & 0 \\ 0 & 0 & e_{15} & \epsilon_{11} & 0 \\ e_{31} & e_{33} & 0 & 0 & \epsilon_{33} \end{pmatrix} \qquad (2.3)$$

In order to obtain a beam theory we assume that the transverse normal stresses are negligible, i.e., that $T_{33} = 0$. We then can solve for S_{33} in (2.1) to obtain $S_{33} = c_{33}^{-1}(e_{33}E_3 - c_{13}S_{11})$. Upon elimination of S_{33}, we obtain a system of the same form, but with $T_{33} = 0$ and S_{33} dropped and the physical constants $[c]$ and $[e]$ renamed. More precisely, in (2.3) we put $(c_{11})_{new} = c_{11} - c_{13}^2/c_{33}$, $(e_{31})_{new} = e_{31} - c_{13}e_{33}/c_{33}$ and $(\epsilon_{33})_{new} = \epsilon_{33} + e_{33}^2/c_{33}$. Once all this is done we are left with a simplified set of constitutive laws:

$$\begin{array}{ll} T_{11} = c_{11}S_{11} - e_{31}E_3 & T_{33} = 0 \\ T_{13} = c_{55}S_{13} - e_{15}E_1 & D_1 = e_{15}S_{13} + \epsilon_{11}E_1 \\ D_3 = e_{31}S_{11} + \epsilon_{33}E_3. & \end{array} \qquad (2.4)$$

In the electrostatic approximation one assumes the existence of an electric potential ϕ such that

$$E = -\nabla\phi. \qquad (2.5)$$

Thus $(E_1, E_3) = -(\phi_x, \phi_z)$. Using (as in Rogacheva [3]) the first three Taylor terms in the z direction we write

$$\phi = \phi^0 + z\phi^1 + \frac{z^2}{2}\phi^2. \qquad (2.6)$$

With Timosenko displacement assumptions we write

$$U_3 = w, \qquad U_1 = v + z\psi$$

where (U_1, U_3) denotes the displacement field, w denotes the transverse displacement (constant throughout the thickness), z denotes the transverse coordinate with $z = 0$ occuring on the centerline, v is the longitudinal displacement of the centerline and ψ represents the rotation. We assume the displacement field is related to the strain tensor via the usual small-strain assumption. Thus

$$S_{11} = v_x + z\psi_x, \qquad S_{13} = \frac{1}{2}(\psi + w_x).$$

The potential energy $\mathcal{P}$ and electrical energy $\mathcal{E}$ are given by

$$\begin{array}{l} \mathcal{P} = \frac{1}{2}\int_0^L \int_{-h/2}^{h/2} c_{11}S_{11}^2 + c_{55}S_{13}^2 - e_{31}S_{11}E_3 \\ \qquad\qquad -e_{15}E_1S_{13}\,dx\,dz \\ \mathcal{E} = \frac{1}{2}\int_0^L \int_{-h/2}^{h/2} e_{15}S_{13}E_1 + \epsilon_{11}E_1^2 + e_{31}S_{11}E_3 \\ \qquad\qquad +\epsilon_{33}E_3^2\,dx\,dz \end{array} \qquad (2.7)$$

As in Tzou [6] it is convenient to introduce the *enthalpy* $\mathcal{H} = \mathcal{P} - \mathcal{E}$. Using the displacement assumptions $\mathcal{H}$ can be written as the sum of three terms; the first purely mechanical, the second electro-mechanical and the third purely electrical:

$$\begin{array}{l} \mathcal{H} = \frac{1}{2}a(v,\psi,w;v,\psi,w) + m(v,\psi,w,\phi^0,\phi^1,\phi^2) \\ \qquad -\frac{1}{2}b(\phi^0,\phi^1,\phi^2;\phi^0,\phi^1,\phi^2) \end{array} \qquad (2.8)$$

where

$$\begin{array}{l} a(v,\psi,w;v,\psi,w) = \int_0^L c_{11}(hv_x^2 + \frac{h^3}{12}\psi_x^2) \\ \qquad +c_{55}(\psi + w_x)^2 h\,dx \\ m(v,\psi,w,\phi^0,\phi^1,\phi^2) = \int_0^L he_{31}v_x\phi^1 + \frac{h^3}{12}e_{31}\phi^2\psi_x \\ \qquad +e_{15}(\psi + w_x)(\phi_x^0 h + \frac{h^3}{24}\phi_x^2)\,dx \\ b(\phi^0,\phi^1,\phi^2;\phi^0,\phi^1,\phi^2) = \int_0^L \epsilon_{33}h((\phi^1)^2 + \frac{h^2}{12}(\phi_x^2)^2) \\ \qquad +h\epsilon_{11}(\phi_x^0 + \frac{h^2}{24}\phi_x^2)^2 + \epsilon_{11}\frac{h^3}{12}(\phi_x^1)^2 + \frac{\epsilon_{11}h^5}{5\cdot 144}(\phi_x^2)^2\,dx. \end{array} \qquad (2.9)$$

The kinetic energy $\mathcal{K}$ can be expressed

$$\begin{array}{ll} \mathcal{K} & = \frac{1}{2}c(v_t,\psi_t,w_t;v_t,\psi_t,w_t) \\ & = \frac{1}{2}\int_0^L \rho hw_t^2 + \rho v_t^2 + \rho\frac{h^3}{12}\psi_t^2\,dx \end{array} \qquad (2.10)$$

2.1.1 Equations of motion:

The variational principle is to minimize the exchange of energy from potential to electrical or kinetic. (See Tzou [6].) With appropriately defined work terms $\mathcal{W}$ included the Lagrangian is given by $\mathcal{L} = \int_0^T \mathcal{K} - \mathcal{H} + \mathcal{W}\,dt$. We set the variation with respect to all "admissible displacements" $\{\hat{v}, \hat{\phi}, \hat{w}, \hat{\phi}^0, \hat{\phi}^1, \hat{\phi}^2\}$ of $\mathcal{L}$ to zero to obtain

$$\begin{array}{l} \int_0^T\{c(v_t,\psi_t,w_t;\hat{v}_t,\hat{\psi}_t,\hat{w}_t) - a(v,\psi,w;\hat{v},\hat{\psi},\hat{w}) \\ +b(\phi^0,\phi^1,\phi^2;\hat{\phi}^0,\hat{\phi}^1,\hat{\phi}^2) - m(v,\psi,w;\hat{\phi}^0,\hat{\phi}^1,\hat{\phi}^2) \\ -m(\hat{v},\hat{\psi},\hat{w};\phi^0,\phi^1,\phi^2) + \int_0^L f_1\hat{v} + m_1\hat{\psi} + f_3\hat{w}\,dx \\ + \int_0^L -q(\hat{\phi}^0 + \frac{h^2}{24}\hat{\phi}^2) - r\hat{\phi}^1\,dx\}dt = 0 \end{array} \qquad (2.11)$$

where f_1, m_1, f_3, q, and r are appropriately defined external force resultants. End forces can also be included in a similar way.

After several integrations by parts of (2.11) we obtain the following six equations of motion

$$\begin{array}{rl} \rho hw_{tt} - c_{55}h(\psi + w_x)_x - e_{15}(h\phi_{xx}^0 + \frac{h^3}{24}\phi_{xx}^2) & = f_3 \\ \frac{\rho h^3}{12}\psi_{tt} + c_{55}h(\psi + w_x) - c_{11}\frac{h^3}{12}\psi_{xx} - \frac{h^3}{12}e_{31}\phi_x^2 & \\ +e_{15}(\phi^0 h + \frac{h^3}{24}\phi_x^2) & = m_1 \\ -\epsilon_{11}h(\phi_{xx}^0 + \frac{h^2}{24}\phi_{xx}^2) + he_{15}(\psi + w_x)_x & = q \\ \epsilon_{33}\frac{h^3}{12}\phi^2 - \epsilon_{11}\frac{h^3}{24}(\phi_{xx}^0 + \frac{h^2}{24}\phi_{xx}^2) - \epsilon_{11}\frac{h^5}{5\cdot 144}\phi_{xx}^2 & \\ -e_{31}\frac{h^3}{12}\psi_x + e_{15}\frac{h^3}{24}(\psi + w_x)_x & = \frac{h^2}{24}q \\ \rho hv_{tt} - hc_{11}v_{xx} - he_{31}\phi_x^1 & = f_1 \\ -\epsilon_{11}\frac{h^3}{12}\phi_{xx}^1 + \epsilon_{33}h\phi^1 - he_{31}v_x & = r \end{array} \qquad (2.12)$$

Note that the first four equation of (2.12) describe bending motions and involve only the variables w, ψ,

ϕ^2 and ϕ^2 while the last two describe stretching motions and involve only v and ϕ^1. The same follows also for the boundary conditions, below.

Many combinations of boundary conditions are possible. For example any of the terms w, ψ, v, ϕ^0, ϕ^1, ϕ^2 may be specified at an end. The dual boundary conditions (describing boundary forces or charges) are respectively,

- $\{c_{55}h(\psi + w_x) + e_{15}(\phi^0_x h + \frac{h^3}{24}\phi^2_x)\}$ (shear),
- $\{\frac{h^3}{12}(c_{11}\psi_x + e_{31}\phi^2)\}$ (bending moment),
- $\{hc_{11}v_x + he_{31}\phi^1\}$ (lateral force),
- $\{\epsilon_{11}h(\phi^0_x + \frac{h^2}{24}\phi^2_x) - e_{15}h(\psi + w_x)\}$ (charge),
- $\{\epsilon_{11}\frac{h^3}{24}(\phi^0_x + \frac{h^2}{24}\phi^2_x) + \epsilon_{11}\frac{h^5}{5\cdot 144}\phi^2_x - \frac{e_{15}h^3}{24}(\psi + w_x)\}$ (2nd charge moment)
- $\{\epsilon_{11}\frac{h^3}{12}\phi^1_x\}$ (1st charge moment).

2.1.2 Elimination of electrical variables:

In the case we are interested in, the only external applied force is the the charge r in the last equation in (2.12). If the electrodes are assumed to be equipotential surfaces that are insolated on the edges, then $r = r(t) \cdot H(x)$ where $H \equiv 1$ on $(0, L)$. The patch system becomes

$$\rho h v_{tt} - hc_{11}v_{xx} - he_{31}\phi^1_x = 0 \quad (2.13)$$

$$-\frac{\epsilon_{11}h^3}{12}\phi^1_{xx} + \epsilon_{33}\phi^1 - he_{31}v_x = r(t)H(x). \quad (2.14)$$

The above are valid for $t > 0$ and $0 < x < L$. The boundary conditions are

$$c_{11}v_x(i,t) + e_{31}\phi^1(i,t) = \phi^1_x(i,t) = 0 \qquad i = 0, L. \quad (2.15)$$

Let $y = P_\delta f$ be a solution to

$$-\delta y_{xx} + y = f; \quad y_x(0) = y_x(L) = 0$$

Then

$$\phi^1 = \frac{e_31}{\epsilon_{33}}P_\delta v_x + \frac{r(t)}{h\epsilon_{33}}P_\delta H(x); \quad \delta = \frac{\epsilon_{11}h^2}{12\epsilon_{33}}. \quad (2.16)$$

Since H is constant, $P_\delta H = H + C$ (C arbitrary). Therefore (2.13) becomes

$$\rho h v_{tt} - hc_{11}v_{xx} - h\frac{e^2_{31}}{\epsilon_{33}}(P_\delta v_x)_x = r(t)\frac{e_{31}}{\epsilon_{33}}\frac{d}{dx}H(x). \quad (2.17)$$

Several remarks are in order.

Remark 1. In (2.17) $\frac{d}{dx}H(x)$ needs to be interpreted in the sense of distributions. In this sense, one substitutes

$$(H(x))_x \equiv (\delta(x) - \delta(L-x))$$

into (2.17) where δ denotes the Dirac delta function. Therefore the effect of a voltage difference on the surfaces of the patch is equivalent to an outward force acting on the edge of the patch.

Remark 2. Noting that P_δ is a positive compact operator which tends to the identity (modulo a constant) when $\delta \to 0$, we see that the effect of the electrical coupling is to stiffen the beam, in fact, using $P_\delta \approx I$ we see from (2.17) that the membrane stiffness c_{11} is approximately increased to $c_{11} + e^2_{31}/\epsilon_{33}$. However this approximation becomes less relevant at higher frequencies of excitation since P_δ is compact.

Remark 3. In the usual approach, the term $\epsilon_{11}h^3\phi^1_{xx}$ in (2.13) never appears since the electrical coupling is eliminated earlier by assuming a stress-free mechanical state. This is essentially the same as putting $\delta = 0$ in (2.17). In that case formal eliminatiion of ϕ^1 results in many of the same observations, however this approach becomes less valid when other types of boundary conditions become relevant on the patch. This becomes especially relevant when analyzing the beam-patch interaction along the edge of the patch.

2.2 Piezoelectric Plates

The derivation of the piezoelectric plates proceeds in a similar fashion as with the piezoelectric beam. In particular, one again finds that the equations decouple into a set of "bending equations" and a set of "stretching equations". Thus since the stretching equations are more relevant to vibration control, we will assume ahead of time that the displacement field $\{U_i\}$, $i = 1, 2, 3$ satisfies

$$U_3 = 0, \qquad U_1 = v_1(x,y), \qquad U_2 = v_2(x,y)$$

in a domain Ω of the $x-y$ plane. Therefore the relevant strains are

$$S_{11} = (v_1)_x =: v_{1,1} \quad S_{22} = (v_2)_y =: v_{2,2} \quad (2.18)$$

$$S_{12} = \frac{1}{2}((v_1)_y + (v_2)_x) =: \frac{1}{2}(v_{1,2} + v_{2,1}). \quad (2.19)$$

Again we assume the electrostatic approximation (2.5) holds and the Taylor approximation (2.6) is valid. But since only the ϕ^1 component of the electric potential couples to the stretching motions, we may assume that

$$\phi = z\phi^1, \text{ and hence } E_3 = -\phi^1$$

The relevant constitutive laws are

$$T_{11} = c_{11}S_{11} + c_{12}S_{22} - e_{31}E_3$$
$$T_{22} = c_{11}S_{22} + c_{12}S_{11} - e_{31}E_3$$
$$T_{12} = c_{66}S_{12} = (c_{11} - c_{12})S_{12}, \quad D_1 = \epsilon_{11}E_1$$
$$D_2 = \epsilon_{11}E_2, \qquad D_3 = e_{31}S_{12} + \epsilon_{33}E_3$$

In terms of Young's modulus E and Poisson's ration μ,

$$\begin{aligned}(c_{11},c_{12},c_{66}) &= (\frac{E}{1-\mu^2},\frac{\mu E}{1-\mu^2},\frac{E}{2(1+\mu)}) \\ &=: (D,\mu D, D\frac{1-\mu}{2}).\end{aligned} \tag{2.20}$$

The expressions for potential energy $\mathcal{P}$ and electric energy $\mathcal{E}$ are then easily obtained. From this, using the strain-displacement equations (2.18)–(2.19), we can calculate the enthalpy $\mathcal{H}$ in terms of the variables $v = (v_1, v_2)$ and ϕ^1. We again separate this into its mechanical, electro-mechanical, and electrical components as follows.

$$\mathcal{H} = \frac{1}{2}a(v;v) + m(v,\phi^1) - \frac{1}{2}b(\phi^1;\phi^1)$$

where

$$\begin{aligned}a(v;\hat{v}) &= Dh\int_\Omega[(v_{1,1}\hat{v}_{1,1} + v_{2,2}\hat{v}_{2,2} + \mu v_{1,1}\hat{v}_{2,2} \\ &\quad +\mu v_{2,2}\hat{v}_{1,1} + \tfrac{1-\mu}{2}(v_{1,2}+v_{2,1})(\hat{v}_{1,2}+\hat{v}_{2,1})]\,d\Omega \\ m(v,\phi^1) &= \int_\Omega[he_{31}(v_{1,1}+v_{2,2})\phi^1]\,d\Omega \\ b(\phi,\hat{\phi}) &= \int_\Omega \tfrac{\epsilon_{11}h^3}{12}(\phi_x\hat{\phi}_x + \phi_y\hat{\phi}_y) + h\epsilon_{33}\phi\hat{\phi}\,d\Omega\end{aligned} \tag{2.21}$$

We have the Green's formula

$$a(v,\hat{v}) = (\mathcal{B}v,\hat{v})_\Gamma - (Lv,\hat{v})_\Omega$$

where $(f,g)_X$ refers to integration over X ($X = \Omega$ or $X = \Gamma = \text{bdry}(\Omega)$), L is the elliptic operator

$$\begin{aligned}(L(v))_1 &= (v_{1,11} + \frac{1-\mu}{2}v_{1,22} + \frac{1+\mu}{2}v_{2,12} \\ (L(v))_2 &= (v_{2,22} + \frac{1-\mu}{2}v_{2,11} + \frac{1+\mu}{2}v_{1,12})\end{aligned} \tag{2.22}$$

and $\mathcal{B}$ is the boundary operator associated with $\mathcal{L}$ (see Lagnese and Lions [2] or Hansen [1]).

2.2.1 Equations of motion: Again the equations of motion are easily calculated by the Lagrangian approach. In the case where the only external force is the surface charge r, setting the variation of the Lagrangian to zero gives

$$\begin{aligned}&\int_0^T\{c(v_t;\hat{v}_t) - a(v;\hat{v}) + b(\phi^1,\hat{\phi}^1) \\ &-m(\hat{v},\phi^1) - m(v,\hat{\phi}^1) - (r,\phi^1)\} = 0\end{aligned} \tag{2.23}$$

If $r = r(t)H(x,y)$, where $H \equiv 1$ on Ω the equations take the form

$$\begin{aligned}&\rho h v_{tt} - hDLv - he_{31}\nabla\phi^1_x = 0 \\ &-\frac{\epsilon_{11}h^3}{12}\Delta\phi^1 + h\epsilon_{33}\phi^1 - he_{31}(v_{1,1}+v_{2,2}) = r(t)H(x).\end{aligned} \tag{2.24}$$

The above are valid for $t > 0$ and $(x,y) \in \Omega$. In the insolated, stress free case the boundary conditions are

$$hD\mathcal{B}v + he_{31}\phi^1(1,1)^T = 0, \quad \phi^1_n = 0,$$

which holds on $\Gamma = \text{bdry}(\Omega)$

2.2.2 Elimination of electrical components: The previous analysis also applies to the plate case. We define P_δ as in (2.16). Then

$$\phi^1 = \frac{e_{31}}{\epsilon_{33}}P_\delta(v_{1,1}+v_{2,2}) + \frac{r(t)}{h\epsilon_{33}}P_\delta H(x); \quad \delta = \frac{\epsilon_{11}h^2}{12\epsilon_{33}}.$$

Since H is constant, $P_\delta H = H + C$ (C arbitrary). Therefore system (2.24) reduces to

$$\rho h v_{tt} - hDLv - h\frac{e_{31}^2}{\epsilon_{33}}\nabla(P_\delta(v_{1,1}+v_{2,2}) = r(t)\frac{e_{31}}{\epsilon_{33}}\nabla H(x). \tag{2.25}$$

In this case, when the forcing term is interpreted in the proper variational sense, it is seen to be concentrated on the boundary and directed in the outward normal direction, much in analogy with the beam case mentioned in Remark 1. As in Remark 2, again the electrical coupling is seen to effectively stiffen plate, although this effect is reduced at high frequencies due to the fact that P_δ is compact.

3 Patch-Plate Coupling

We now turn to the problem of modeling a localized patch on a plate or beam. Although this is a well researched topic, a consistent and aesthetic variational formulation of the problem with the higher order electrical terms included apparently does not exist. This is part of the goal of this work.

Again we introduce all the ideas for the case of a localized patch on a beam.

3.1 Modeling of a patch on a beam

We consider a beam of thickness $2h_0$ that occupies at equilibrium the interval $\Omega = (0,L)$ on which a piezoelectric patch of thickness h_1 is mounted on the surface on a subinterval $\omega = (a,b)$ of Ω. To further simplify this discussion, let us also assume there is another patch on the opposite surface that is run out of phase with the upper patch, so that we only need to consider beam motions that are antisymmetric with respect to the centerline of the beam. (An extra equation is needed without this assumption, but everything here can be extended to the case of a single patch.)

Assume that the beam has material parameters subscripted with 0's (density ρ_0, thickness h_0, etc.) and the corresponding quantities for the patch are subscripted with 1's. Our initial modeling of the plate-patch system is based upon the multilayer theory of [1]. To obtain the model that contains the first order effects of the input voltage, we begin by writing out all the energy terms for both the plate and the patch. We assume that h_1 is small compared to h_0 so that all mechanical terms of order h_1^2 may be dropped.

One can see from (2.9) that the only energy terms in the patch of linear order in h_1 are the stretching and shear terms (and associated electrical terms). However since the patch is assumed to be very thin the effect of shear is negligible. Hence we may assume that the patch is reasonably well described by including only the lateral displacement v and the charge moment ϕ. (Similar to the derivation of (2.13)–(2.14).) If we model the beam by Timoshenko assumptions, the state of the beam is described by the rotation angle ψ and the transverse displacement w.

The energy terms can be calculated as the sum of energies of the beam (in terms of w, ψ) and those of the patch (in terms of v, ϕ). Since there is no slipping of the patch we also have the geometric constraint $v = h_0\psi$ to be substituted in the energies.

We then can calculate the enthalpy $\mathcal{H}$ and the kinetic energy of the coupled system in terms of w, ψ and ϕ as

$$\mathcal{H} = \frac{1}{2}a(w,\psi;w,\psi) + m(\psi,w,\phi) - \frac{1}{2}b(\phi;\phi)$$

where

$$\begin{aligned} a(w,\psi;\hat{w},\hat{\psi}) &= \int_\omega D_1 h_1 h_0^2 \psi_x \hat{\psi}_x \, dx + \int_\Omega [4h_0^3 D_0 \psi_x \hat{\psi}_x \\ &\quad + G_0 h_0 (\psi_x + w_x)(\hat{\psi}_x + \hat{w}_x) \, dx \\ m(\psi,w,\phi) &= \int_\omega h_1 e_{31} (h_0\psi)_x \phi \, dx \\ b(\phi,\hat{\phi}) &= \int_\omega \epsilon_{11} \tfrac{h^3}{12} \phi_x \hat{\phi}_x + \epsilon_{33} h \phi \hat{\phi} \, dx. \end{aligned} \tag{3.26}$$

The kinetic energy $\mathcal{K}$ can be expressed

$$\begin{aligned} \mathcal{K} &= \tfrac{1}{2} c(w_t,\psi_t;w_t,\psi_t) \\ &= \tfrac{1}{2} \int_\Omega \rho h w_t^2 + \rho \tfrac{h^3}{12} \psi_t^2 \, dx + \int_\omega \rho_1 h_1 w_t^2 \, dx \end{aligned} \tag{3.27}$$

3.1.1 Equations of motion: When the work term for the surface charge r is included in the work term $\mathcal{W}$ the Lagrangian is given by $\mathcal{L} = \int_0^T \mathcal{K} - \mathcal{H} + \mathcal{W}\,dt$. To obtain the equations of motion, the variation *with respect to all "admissible displacements"* of $\mathcal{L}$ is set to 0. For this we apply the usual hard constraints to the test functions on the boundary of Ω, while on bdry(ω) we apply the following natural conditions:

$$\hat{w} \text{ is continuous on bdry}(\omega) \tag{3.28}$$

$$\hat{\psi} \text{ is continuous on bdry}(\omega) \tag{3.29}$$

We then obtain

$$\begin{aligned} \rho(x) w_{tt} - G_0 h_0 (\psi + w_x)_x &= 0 \\ \alpha(x)\psi_{tt} - D(x)\psi_{xx} + G_0 h_0(\psi + w_x) - e(x)\phi_x &= 0 \\ -\tfrac{\epsilon_{11} h_1^3}{12} \phi_{xx} + h_1 \epsilon_{33} \phi - e(x)\psi_x &= r \end{aligned} \tag{3.30}$$

where ρ, α D and e are the piecewise constant functions defined by

$$\rho = \rho_0 h_0 + \chi_\omega \rho_1 h_1 \tag{3.31}$$

$$\alpha = \frac{1}{3}\rho_0 h_0^3 + \chi_\omega \rho_1 h_1 h_0^2 \tag{3.32}$$

$$D = 4h_0^3 D_0 + \chi_\omega h_1 h_0^2 D_1 \tag{3.33}$$

$$e = \chi_\omega h_1 h_0 e_{31}. \tag{3.34}$$

The natural boundary conditions associated with system (3.30) are easily obtained by the variational approach. The boundary conditions on bdry(Ω) can be any of the usual ones for the Timoshenko beam, for example in the clamped case they are

$$w = \psi = 0 \qquad \text{on bdry}(\Omega) \times \mathbb{R}^+.$$

Assuming that the edges of the patch are unstressed and electrically insolated the boundary conditions obtained through the variational method take the form of *jump conditions* on the edge of the domain of the patch. For our beam with patch localized on bdry(ω) they are

$$\phi_x = 0 \qquad \text{on bdry}(\omega) \tag{3.35}$$

$$w_x \text{ continuous} \qquad \text{on bdry}(\omega) \tag{3.36}$$

$$D(x)\psi_x + e(x)\phi \text{ continuous} \qquad \text{on bdry}(\omega) \tag{3.37}$$

(In (3.36)–(3.37) we use the convention that a function only defined on ω is extended by the value 0 to all of Ω.)

We remark that the system (3.30) is defined on the domain $\Omega - \text{bdry}(\omega)$, hence the derivatives are defined in a natural sense. Note also that with the electrical coupling ignored system (3.30) reduces to the Timoshenko beam system on each component of the domain.

3.1.2 Elimination of electrical components: Essentially the same conclusions as mentioned in Remarks 1,2,3 remain valid for the coupled system. Namely, the electrical coupling can be eliminated as in (**??**) via the operator P_δ and one obtains a reduced system involving stiffer mechanical components and an input that is proportional to the derivative of a step function. If we put $\delta = 0$ (so that the small term $-h_1^3\epsilon_{11}\phi_{xx}/12$ is dropped) one then obtains in place of (3.30) the following:

$$\begin{aligned} &\rho(x)w_{tt} - G_0 h_0 (\psi + w_x)_x = 0 \\ &\alpha(x)\psi_{tt} - (D(x) + \tfrac{e(x)^2}{\epsilon_{33} h_1})\psi_{xx} \\ &\qquad + G_0 h_0(\psi + w_x) = \{\tfrac{1}{\epsilon_{33} h_1} \tfrac{d}{dx} e(x)\} r(t) \end{aligned} \tag{3.38}$$

Thus the control enters multiplied by the derivative of a step function, as a forcing term in the rotation angle. This, in fact has been one of the difficulties in the mathematical analysis of these models, in that the control enters in such a "rough" fashion. This problem becomes even more pronounced when the the underlying beam model is the Euler-Bernoulli model, (or Kirchhoff plate) since Kirchhoff hypothesis then $\psi = w_x$ (or ∇w in the plate case) and hence one finds that control enters the w equation as the second derivative of a step function.

3.2 Modeling of a patch on a plate.

We conclude with a brief discussion of the problem of modeling a patch on a Reissner-Mindlin plate. Essentially one can follow in analogy all the steps used in the derivation of beam-patch system with the obvious notational changes. In particular ω is a subdomain of Ω which is some region in the $x-y$ plane.

We obtain for the equations of motion

$$\begin{aligned} &\rho(x)w_{tt} - \operatorname{div} G_0 h_0(\psi + \nabla w) &&= 0 \\ &\alpha(x)\psi_{tt} - \tilde{L}\psi + G_0 h_0(\psi + \nabla w) - e(x)\nabla\phi &&= 0 \\ &-\tfrac{\epsilon_{11} h_1^3}{12}\Delta\phi + h_1 \epsilon_{33}\phi - e(x)(\psi_{1,1} + \psi_{2,2}) &&= r \end{aligned} \tag{3.39}$$

where ρ, α and e are the piecewise constant functions defined by (3.31), (3.32), (3.34) and

$$\tilde{L} = 4h_0^3 D_0 L_0 + \chi_\omega h_1 h_0^2 D_1 L_1$$

where, for $i = 0, 1$, L_i is defined by (2.22), but with Poisson's ration μ replaced by μ_i (the Poisson's ratio of layer i).

The natural boundary conditions associated with system (3.39) consist of the usual ones on bdry(Ω) for the Reissner-Mindlin system (see Lagnese and Lions [2]), along with the continuity conditions (from (3.28)–(3.29))

$$w \text{ is continuous on bdry}(\omega) \tag{3.40}$$
$$\psi \text{ is continuous on bdry}(\omega) \tag{3.41}$$

and force balance conditions

$$\nabla w \cdot n \text{ continuous} \quad \text{on bdry}(\omega) \tag{3.42}$$
$$[\tilde{\mathcal{B}}\psi + e(x)(1,1)^T\phi] \text{ continuous} \quad \text{on bdry}(\omega) \tag{3.43}$$

where $\tilde{B}$ is the boundary operator associated with $\tilde{L}$. Equations (3.42)-(3.43) need some explanation. In (3.42) we mean that ∇w is continuous on bdry(ω) in directions normal to bdry(ω). In (3.43) the operator $\tilde{B}$, which involves the normal vector n, should be continuous from each side of ω, using the same vector normal vector n in each case.

In addition the electrical boundary conditions need to be specified. If the electrodes are assumed to be insolated on the edges then $\phi_n = 0$ on bdry(ω) is appropriate. In this case the electrical components may be eliminated in the same way that they were for the beam with a patch. One then obtains a coupled system similar to involving the compact operator P_δ. Setting $\delta = 0$ then leads to a system similar to (3.38).

4 Conclusions

We have seen that by retaining the higher order electrical coupling, and applying the variational approach, one not only obtains well-posed system of partial differantial equations to model piezoelectric patch-plate interactions in which jump conditions arise as a consequence of the continuity conditions in the variation of the Lagrangian. An advantage of keeping the higher-order coupling is that this allows for analysis in the case of other boundary conditions that might be imposed on the boundary of the patch.

Secondly, we mention although no mention of precise regularity or well-posedness properties appears in this article, due to the variational formulation of the problems, the proper foundation has been laid to address these issues.

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