EXACT CONTROLLABILITY OF A BEAM IN AN INCOMPRESSIBLE INVISCID FLUID

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Abstract. It is well known that an Euler-Bernoulli beam may be exactly controlled with a single control acting on an end of the beam. In this article we show that for certain boundary conditions, the same result holds for a beam that is surrounded by an incompressible, inviscid fluid with a sufficiently small density. The proof involves reducing the control problem to a moment problem and using compactness properties of the Neumann to Dirichlet map for the Laplacian operator to obtain the needed estimates.

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1. Introduction. It is well-known that an Euler-Bernoulli beam may be exactly controlled by a single control acting at one end of the beam. However, unless the beam is in a vacuum, there is usually some amount of influence that the surrounding medium will impose upon the beam. Thus it is natural to ask whether the controllability properties of a beam (or other elastic material) are preserved when its motion is coupled with that of a surrounding fluid.

Banks, et. al. [2] proposed a model describing the coupling between the acoustic pressure in a rectangular cavity and the vibrations of an Euler-Bernoulli beam positioned along one edge of the rectangle. In that problem, the motivation was to utilize controllers placed along the beam to stabilize the acoustic pressure inside the cavity. Although much progress has been made on this problem (see [3] for well-posedness, and references therein for stabilization of finite dimensional approximations; also see Micu and Zuazua [8, 9] for *non-uniform* stabilization results) it is impossible to *exactly* control the state of the system by controllers that are active only along one edge of the rectangle. This is due to the presence of "trapped rays" (or echos) (see Bardos, Lebeau, Rauch, [1]).

In this article we investigate the controllability results one can obtain for systems similar to the one in [2], however, for the case of an incompressible fluid. In the incompressible case the fluid motion is described by a potential equation, and hence there are no obstructions to controllability associated with "trapped rays". We show that at least for certain boundary conditions, the exact controllability properties of an Euler-Bernoulli beam (and other elastic models) are retained when the beam is surrounded by an incompressible fluid of sufficiently small density.

1.1. Problem Statement. At equilibrium we assume that a still, incompressible, inviscid fluid occupies the region Ω of the x–y plane with Lipschitz boundary Γ. We assume that Γ consists of a *inflexible* part Γ_1 and and a *flexible* part Γ_0. If Γ_0 is surrounded on both sides by the fluid, then $\Gamma_0 = \Gamma_0^+ \cup \Gamma_0^-$ where Γ_0^+ and Γ_0^- represent the upper and lower surfaces of the beam, respectively. For definiteness we suppose that

$$\Gamma_0^{\pm} = \{(x, \pm h/2) : 0 < x < 1\}$$

for some $h > 0$. Some possible domains are illustrated in Figures 1 and 2.

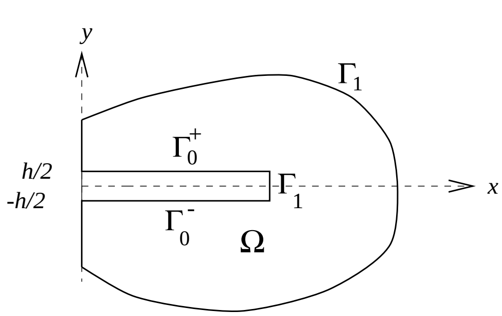

Fig. 1.

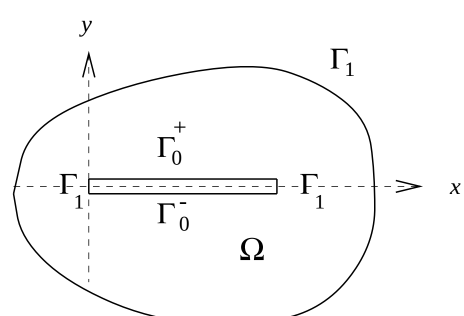

Fig. 2.

The transverse displacement $w(x,t)$ (the displacement in the y direction) satisfies the Euler-Bernoulli beam equation:

$$m\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} = F \qquad \text{on } (0,1) \times \mathbb{R}^+, \tag{1.1}$$

where m denotes the density of the beam, EI denotes the stiffness of the beam and F denotes the force exerted by the fluid on the beam, all in per length units. The force F in (1.1) is the difference in the fluid pressures on the top and bottom of the beam, i.e.,

$$F(x,t) = -p(x, h/2, t) + p(x, -h/2, t) \quad \text{on } (0,1) \times \mathbb{R}^+. \tag{1.2}$$

The fluid we consider is modeled from the theory of ideal fluids, see e.g., [6]. The linearization (about a still state) of the Euler system describing the motion of an ideal fluid is

$$\rho \frac{\partial \mathbf{q}}{\partial t} + \nabla p = 0, \qquad \operatorname{div} \mathbf{q} = 0,$$

where $\mathbf{q} = \mathbf{q}(x, y, t)$ is the velocity field, $p = p(x, y, t)$ is the pressure, and ρ (assumed to be constant) denotes the density per area of the fluid. (We regard the fluid as a two dimensional medium.) Taking the divergence of the first equation gives

$$\Delta p = 0 \qquad \text{in } \Omega \times \mathbb{R}^+ \tag{1.3}$$

and matching the normal component of the fluid acceleration to the acceleration of the boundary leads to the following boundary conditions ($\mathbf{n}$ is the outward unit normal to Γ).

$$\frac{\partial p}{\partial \mathbf{n}} = 0 \qquad \text{on } \Gamma_1 \times \mathbb{R}^+ \tag{1.4}$$

$$\frac{\partial p}{\partial y}(x, -h/2, t) = \frac{\partial p}{\partial y}(x, h/2, t) = -\rho \frac{\partial^2 w}{\partial t^2}(x,t) \qquad \text{on } (0,1) \times \mathbb{R}^+. \tag{1.5}$$

In the above formulation, as in [2], the domain of the fluid is approximated by the fixed domain Ω to avoid nonlinearities.

Remark 1.1. The above model (1.1)–(1.5) becomes a structural-acoustic system when (1.3) is replaced by the wave equation:

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} - \Delta p = 0.$$

Consequently, the incompressible model can be viewed as the limiting case of a structural-acoustic model as the wave speed tends to infinity. The validity of the incompressibility assumption depends upon the particular application. One application where (1.1)–(1.5) has been widely used is in modeling vibrations of the cochlea within the inner ear. (See Lighthill [7] for a thorough discussion.)

Remark 1.2. If the fluid is only on one side of the beam then (1.2) is modified accordingly. However, the resulting mathematical problem is fundamentally different in nature since in that case the compressibility of the fluid imposes an additional constraint upon the possible beam motions that is not imposed when the same fluid body lies on both sides. We restrict our interest in this article to the situation in (1.2).

1.2 Main Results. Let

$$\mathcal{V} = \{f \in H^2(0,1) : f'(0) = f'(1) = 0\}, \qquad \mathcal{H} = L^2(0,1).$$

We will mainly be concerned with the following set of boundary conditions for the beam.

$$\frac{\partial w}{\partial x}(0,t) = \frac{\partial w}{\partial x}(1,t) = \frac{\partial^3 w}{\partial x^3}(1,t) = 0; \quad EI\frac{\partial^3 w}{\partial x^3}(0,t) = f(t), \quad t > 0. \tag{1.6}$$

Initial conditions for the beam are of the form

$$\{w, \partial w/\partial t\}\big|_{t=0} = \{w^0, w^1\} \in \mathcal{V} \times \mathcal{H}. \tag{1.7}$$

The boundary conditions in (1.6) can be viewed as "sliding clamps" at each end. The clamp on the right end moves freely in the transverse direction (we are considering small motions) while a transverse force $f(t)$ is applied at the left end.

Concerning the control problem (1.1), (1.6), (1.7), without the fluid coupling (i.e., with $F \equiv 0$), it is well known that given any initial state $\{w^0, w^1\} \in \mathcal{V} \times \mathcal{H}$, and any $f \in L^2(0,\infty)$, there is a unique solution w which satisfies

$$w \in C([0,\infty); \mathcal{V}) \cap C^1([0,\infty); \mathcal{H}). \tag{1.8}$$

Furthermore the system is *exactly controllable on* $\mathcal{V} \times \mathcal{H}$ *in any time* $T > 0$; that is, given any initial state $\{w^0, w^1\} \in \mathcal{V} \times \mathcal{H}$, any terminal state $\{v^0, v^1\} \in \mathcal{V} \times \mathcal{H}$ and any $T > 0$ there exists $f \in L^2(0,T)$ for which

$$\{w(T), \frac{\partial w}{\partial t}(T)\} = \{v^0, v^1\}. \tag{1.9}$$

The main result of this article are the following.

Theorem 1.1. *Given any* $\{w^0, w^1\} \in \mathcal{V} \times \mathcal{H}$ *and* $f \in L^2(0,\infty)$ *the system (1.1)–(1.7) has a unique solution* w *which satisfies (1.8). Furthermore there exists* $\rho_0 > 0$ *such that if* $0 \le \rho < \rho_0$ *then (1.1)–(1.7) is exactly controllable on* $\mathcal{V} \times \mathcal{H}$ *in any time* $T > 0$*, i.e., given any* $\{w^0, w^1\}$*,* $\{v^0, v^1\} \in \mathcal{V} \times \mathcal{H}$ *there exists* $f \in L^2(0,T)$ *for which the solution to (1.1)–(1.7) satisfies (1.9).*

The solution w in Theorem 1.1 is the unique solution of an equivalent variational formulation of (1.1)–(1.7) in which the pressure p has been eliminated. (See Section 4.)

Remark 1.3. The regularity in Theorem 1.1 is optimal over the class of controls in $L^2(0,T)$. Furthermore the controllability result is optimal in the sense that the regularity space $\mathcal{V} \times \mathcal{H}$ is the same as the controllability space.

Remark 1.4. By using part (2) of Lemma 3.4 (with $l = 1$) and simple estimates, we know that ρ_0 in Theorem 1.1 may be taken to be $.01m/c_0$, where c_0 is the constant defined in Lemma 3.1. (The constant c_0 depends only on the domain Ω.) We do not know what the largest interval $[0, \rho_{\mathrm{opt}})$ of exact controllability is, or if it is finite.

Our approach to the proof of Theorem 1.1 is to reduce the control problem to a moment problem. (See [12, 13] for some early applications of this method). A necessary condition for exact controllability is that the eigenvalues of the system possess a *uniform asymptotic separation.* In the case $\rho = 0$ the eigenvalues are known exactly. For $\rho > 0$ we are able to show that the amount that the eigenvalues are perturbed by the fluid is proportional to ρ and to a certain norm of the "Neumann to Dirichlet map" for the Laplacian (see Section 2). Thus the problem of proving eigenvalue separation reduces to proving certain compactness properties of the Neumann to Dirichlet map. In Section 2 we show that this map is continuous from L^2 to H^1 and this is precisely what is required in the case of boundary conditions (1.6) to prove eigenvalue separation.

The methods used in this paper are valid for certain other types of boundary conditions and also in situations where other dynamics are present on the flexible boundary besides the Euler-Bernoulli beam. In particular, exact controllability results also hold when the Euler-Bernoulli beam is replaced by either a string equation or a Rayleigh beam. These extensions (and some limitations of our approach) are discussed in Section 5.

2. Existence, Uniqueness, Regularity.

In this section we prove the well-posedness of the beam-fluid system.

2.1. Regularity of fluid pressure on beam. We first discuss some known properties of the Neumann problem for the Laplacian.

Consider the following Neumann problem.

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega \\ \frac{\partial\phi}{\partial\mathbf{n}} = f & \text{on } \Gamma. \end{cases} \tag{2.1}$$

The solvability condition for (2.1) is

$$\int_\Gamma f\, d\Gamma = 0. \tag{2.2}$$

It is well known that when Γ and f are regular, and f satisfies (2.2), there exist classical solutions to (2.1) which are unique up to an arbitrary constant. In this case it follows that solutions to (2.1) are unique under the additional condition that

$$\int_\Omega \phi\, d\Omega = 0. \tag{2.3}$$

However if Γ is only Lipschitz, the classical regularity theory for elliptic operators does not necessarily apply; see [5] for example. We therefore consider variational solutions.

Let

$$c(\phi, \psi) = \int_\Omega \nabla\phi \cdot \nabla\bar{\psi}, \qquad (u, v)_\Gamma = \int_\Gamma u\bar{v}\, d\Gamma.$$

and $\tau : H^1(\Omega) \to H^{1/2}(\Gamma)$ denote the trace operator $\tau\phi = \phi|_\Gamma$. By formally applying Green's theorem to (2.1) one obtains the following variational form of (2.1):

$$c(\phi, \psi) = (f, \tau\psi)_\Gamma \qquad \forall\psi \in H^1(\Omega). \tag{2.4}$$

For $s \in [0,1]$ and $U =$ either Ω or Γ, define

$$\tilde{H}^s(U) = \{v \in H^s(U) : \int_U v\, dU = 0\},$$

where $H^s(U)$ denotes the usual Sobolev space of order s on U. For $s \in [-1,0]$ define $\tilde{H}^s(\Gamma) = \tilde{H}^{-s}(\Gamma)'$, i.e., by duality with respect to $\tilde{H}^0(\Gamma)$. Implicit is the identification $\tilde{H}^0(\Gamma) = \tilde{H}^0(\Gamma)'$. The form c is sesquilinear, conjugate-symmetric and continuous on $\tilde{H}^1(\Omega) \times \tilde{H}^1(\Omega)$. Furthermore c is coercive on $\tilde{H}^1(\Omega)$, i.e., for some $\delta > 0$,

$$c(u,u) \geq \delta \|u\|^2_{H^1(\Omega)}, \qquad \forall\, u \in \tilde{H}^1(\Omega). \tag{2.5}$$

Since $\tilde{H}^1(\Omega)$ is densely and compactly embedded in $\tilde{H}^0(\Omega)$ the Lax-Milgram theorem provides the existence and uniqueness of a solution ϕ to the problem

$$\begin{cases} \phi \in \tilde{H}^1(\Omega), \\ c(\phi,\psi) = l(\psi) \qquad \forall \psi \in \tilde{H}^1(\Omega) \end{cases} \tag{2.6}$$

where l is a given element of $\tilde{H}^1(\Omega)'$ (the dual space relative to $\tilde{H}^0$). By the trace theorem, if $f \in L^2(\Gamma)$ the form $l_f(\psi) = (f, \tau\psi)_\Gamma$ defines an element of $\tilde{H}^{-1}(\Omega)$. In fact, extension by continuity shows that l_f remains an element of $\tilde{H}^{-1}(\Omega)$ for any $f \in H^{-1/2}(\Gamma)$. If in addition (2.2) holds, (more precisely, if $f \in \tilde{H}^{-1/2}(\Gamma)$) then ϕ satisfies (2.6) (with $l = l_f$) if and only if (2.4) is satisfied. We therefore have that given any $f \in \tilde{H}^{-1/2}(\Gamma)$ there is a unique solution $\phi \in \tilde{H}^1(\Omega)$ to (2.4).

Let G denote the solution operator to (2.6), i.e.,

$$\phi = Gf \Leftrightarrow \begin{cases} \phi \in \tilde{H}^1(\Omega) \\ c(\phi,\psi) = (f, \tau\psi)_\Gamma, \qquad \forall \psi \in \tilde{H}^1(\Omega). \end{cases}$$

By the preceding discussion $G : \tilde{H}^{-1/2}(\Gamma) \to \tilde{H}^1(\Omega)$ is continuous.

Now define

$$\Lambda f = \tau G f - \frac{1}{\mu(\Gamma)} \int_\Gamma \tau G f\, d\Gamma,$$

where $\mu(\Gamma)$ is the measure of Γ. Λ is called the *Neumann to Dirichlet map* and has been studied in the literature (e.g., [4, 14]). We have, due to the continuity of G and τ,

$$\Lambda : \tilde{H}^{-1/2}(\Gamma) \to \tilde{H}^{1/2}(\Gamma) \;\text{ continuously.} \tag{2.7}$$

Λ is easily seen to be self-adjoint:

$$\langle \Lambda f, g\rangle_\Gamma = c(Gf, Gg) = \langle f, \Lambda g\rangle_\Gamma \qquad \forall\, f, g \in \tilde{H}^{-1/2}(\Gamma). \tag{2.8}$$

($\langle \cdot\,,\cdot\rangle_\Gamma$ denotes the duality pairing which coincides with $(\cdot\,,\cdot)_\Gamma$ when both arguments are in $\tilde{H}^0$.)

Nečas [10] showed that τG remains continuous from $\tilde{H}^0(\Gamma)$ to $H^1(\Gamma)$. It follows that Λ remains continuous as a mapping from $\tilde{H}^0(\Gamma) \to \tilde{H}^1(\Gamma)$. Therefore we have the following result.

Theorem 2.1. *The operator* $\Lambda : \tilde{H}^0(\Gamma) \to \tilde{H}^1(\Gamma)$ *continuously and satisfies for some* $C > 0$

$$\|\Lambda g\|_{H^1(\Gamma)} \leq C\|g\|_{L^2(\Gamma)} \qquad \forall\, g \in \tilde{H}^0(\Gamma).$$

From (1.3)–(1.5) we see that the Neumann problem we must solve to obtain the fluid pressure on the beam is of the following type:

$$\begin{cases} \Delta p = 0 & \text{in } \Omega \\ \frac{\partial p}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_1 \\ \frac{\partial p}{\partial \mathbf{n}} = g & \text{on } \Gamma_0^+ = \{(x, h/2) \,|\, 0 < x < 1\} \\ \frac{\partial p}{\partial \mathbf{n}} = -g & \text{on } \Gamma_0^- = \{(x, -h/2) \,|\, 0 < x < 1\}. \end{cases} \tag{2.9}$$

In (2.9) note that the solvability condition (2.2) will always be satisfied for any $g \in L^2(0,1)$. Therefore by the previous discussion and Theorem 2.1, (2.9) has a unique variational solution p_g such that $\tau p_g \in \tilde{H}^1(\Gamma)$. Define $\Lambda_0 : L^2(0,1) \to H^1(0,1)$, by

$$\Lambda_0 g = (\Lambda\tilde{g})|_{\Gamma_0^+} - (\Lambda\tilde{g})|_{\Gamma_0^-}\,; \qquad \tilde{g} = \begin{cases} g & \text{on } \Gamma_0^+ \\ -g & \text{on } \Gamma_0^- \\ 0 & \text{on } \Gamma_1. \end{cases} \tag{2.10}$$

Proposition 2.2. *The operator* $\Lambda_0 : L^2(0,1) \to H^1(0,1)$ *is continuous and satisfies for some* $C > 0$

$$\|\Lambda_0 g\|_{H^1(0,1)} \leq C\|g\|_{L^2(0,1)}. \tag{2.11}$$

Furthermore, Λ_0 *is a positive and self-adjoint operator on* $L^2(0,1)$.

Proof. Estimate (2.11) follows from (2.10) and Theorem 2.1. If $f, g \in L^2(0,1)$ then using (2.8), (2.10) one obtains

$$\begin{aligned} \int_0^1 (\Lambda_0 g) f \, d\ell &= \int_0^1 (\Lambda\tilde{g}|_{\Gamma_0^+} - \Lambda\tilde{g}|_{\Gamma_0^-}) f \, d\ell = \int_{\Gamma_0^+} (\Lambda\tilde{g})\tilde{f}\, d\Gamma + \int_{\Gamma_0^-} (\Lambda\tilde{g})\tilde{f}\, d\Gamma \\ &= \int_\Gamma (\Lambda\tilde{g})\tilde{f}\, d\Gamma = \int_\Gamma \tilde{g}(\Lambda\tilde{f})\, d\Gamma = \int_0^1 g(\Lambda_0 f)\, d\ell. \end{aligned}$$

Thus Λ_0 is self-adjoint. Since $c(\cdot\,,\cdot)$ satisfies (2.5), the above equalities also show that

$$\int_0^1 (\Lambda_0 f)\bar{f}\, d\ell = c(G\tilde{f}, G\tilde{f}) \geq \delta\|G\tilde{f}\|^2_{H^1(\Omega)} > 0, \qquad \forall f \in L^2(0,1),\ f \neq 0$$

Thus Λ_0 is positive. $\square$

Remark 2.1. For a given $g \in L^2(0,1)$, the solution p to (2.9) is only determined up to a constant, but $\Lambda_0 g$ is uniquely determined. Thus the condition $\int_\Omega p\, d\Omega = 0$ used in defining Λ (and hence also in defining Λ_0) in (2.1) provides uniqueness for p, but the forces acting on the beam are uniquely determined, independent of the number that $\int_\Omega p\, d\Omega$ is set equal to.

2.3. Regularity of coupled system. We can eliminate p from (1.1), (1.2) by noting that the force term F in (1.2) reduces to

$$F = -\rho \Lambda_0 \frac{\partial^2 w}{\partial t^2}.$$

Thus (1.1)–(1.5) can be rewritten as

$$w_{tt} + \frac{EI}{m} w_{xxxx} = -\mu \Lambda_0 w_{tt} \qquad \text{on } (0,1) \times \mathbb{R}^+, \tag{2.12}$$

where

$$\mu = \frac{\rho}{m}$$

and the subscripted variables represent differentiation with respect to that variable. (Henceforth we adopt this notation.)

We first consider the case of homogeneous boundary conditions:

$$w_x(0,t) = w_x(1,t) = w_{xxx}(0,t) = w_{xxx}(1,t) = 0. \tag{2.13}$$

Define the sesquilinear forms

$$a(u,v) = \int_0^1 \frac{EI}{m} u_{xx} \bar{v}_{xx}\, dx, \qquad b(u,v) = \int_0^1 u\bar{v} + \mu(\Lambda_0 u)\bar{v}\, dx.$$

In the variational formulation of (2.12), (2.13), we seek a function $w \in C([0,\infty);\mathcal{V}) \cap C^1([0,\infty);\mathcal{H})$ satisfying

$$\frac{d}{dt} b(w_t, \hat{w}) + a(w, \hat{w}) = 0, \qquad \forall\, \hat{w} \in \mathcal{V}, \tag{2.14}$$

where the differential equation is understood in the sense of distributions on $[0,\infty)$.

By Proposition 2.2, the form b is conjugate-symmetric and continuous on $\mathcal{H} \times \mathcal{H}$. Furthermore since Λ_0 is nonnegative,

$$b(v,v) \geq \|v\|_{\mathcal{H}}^2. \tag{2.15}$$

Likewise a is symmetric and continuous on $\mathcal{V} \times \mathcal{V}$, and for any $\lambda > 0$ there is a positive $\gamma = \gamma(\lambda)$ for which

$$a(v,v) + \lambda b(v,v) \geq \gamma \|v\|_{H^2(0,1)}^2, \qquad \forall\, v \in \mathcal{V}. \tag{2.16}$$

Let us make the identification $\mathcal{H} = \mathcal{H}'$, so that the we have the dense and continuous embeddings

$$\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'.$$

By the Lax-Milgram Theorem there exist an isomorphism $B : \mathcal{H} \to \mathcal{H}$ and a continuous mapping $A : \mathcal{V} \to \mathcal{V}'$ such that

$$\begin{aligned} b(w,\hat{w}) &= (Bw, \hat{w}), \qquad &\forall w, \hat{w} \in \mathcal{H} \\ a(w,\hat{w}) &= \langle Aw, \hat{w} \rangle, \qquad &\forall w, \hat{w} \in \mathcal{V} \end{aligned}$$

where $A + \lambda B : \mathcal{V} \to \mathcal{V}'$ is an isomorphism for any $\lambda > 0$. (In the above equalities, $(\cdot\,,\cdot)$ is the inner product on $\mathcal{H} = L^2(0,1)$ and $\langle\cdot\,,\cdot\rangle$ represents the duality pairing between $\mathcal{V}$ and $\mathcal{V}'$ that coincides with $(\cdot\,,\cdot)$ when both arguments are in $\mathcal{H}$.)

We can write (2.14) in an operational form as

$$\frac{d}{dt} B w_t + A w = 0 \qquad \text{in } \mathcal{V}' \tag{2.17}$$

The first order form of (2.17) can be rewritten

$$\mathcal{B} W_t = \mathcal{A} W, \tag{2.18}$$

where

$$W = \begin{pmatrix} w \\ w_t \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}, \qquad \mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}.$$

We set

$$\mathcal{D}(A) = \{ f \in H^4(0,1) \cap \mathcal{V} : f_{xxx}(0) = f_{xxx}(1) = 0 \}.$$

It is easy to show that for any $\lambda > 0$, $A + \lambda B : \mathcal{D}(A)$ onto $\mathcal{H}$. Define

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times \mathcal{V}$$

and note that $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{V} \times \mathcal{H}$ is continuous and $\mathcal{B} : \mathcal{V} \times \mathcal{H} \to \mathcal{V} \times \mathcal{H}$ is an isomorphism. Therefore $\mathcal{B}^{-1}\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{V} \times \mathcal{H}$ is continuous. In particular, $\mathcal{B}^{-1}\mathcal{A}$ is densely defined as an operator on $\mathcal{V} \times \mathcal{H}$.

Define the sesquilinear form e on $\mathcal{V} \times \mathcal{H}$ by

$$e(\{u_1, u_2\}, \{v_1, v_2\}) = a(u_1, v_1) + b(u_2, v_2).$$

For all $\{u_1, u_2\}, \{v_1, v_2\} \in \mathcal{D}(\mathcal{A})$ we have

$$\begin{aligned} e(\mathcal{B}^{-1}\mathcal{A}\{u_1, u_2\}, \{v_1, v_2\}) &= a(u_2, v_1) - b(B^{-1}(Au_1), v_2) \\ &= a(u_2, v_1) - (Au_1, v_2) \\ &= a(u_2, v_1) - a(u_1, v_2) \\ &= -[a(v_2, u_1) - (Av_1, u_2)] \\ &= -e(\{u_1, u_2\}, \mathcal{B}^{-1}\mathcal{A}\{v_1, v_2\}), \end{aligned}$$

which shows that $\mathcal{B}^{-1}\mathcal{A}$ is skew-adjoint with respect to $e(\cdot\,,\cdot)$. Define the spaces

$$\begin{aligned} &E = \tilde{\mathcal{V}} \times \tilde{\mathcal{H}}; \quad \tilde{\mathcal{V}} = \{u \in \mathcal{V} : b(u, 1) = 0\}, \quad \tilde{\mathcal{H}} = \{u \in \mathcal{H} : b(u, 1) = 0\}, \\ &Z = \mathcal{V}_0 \times \mathcal{H}_0; \quad \mathcal{V}_0 = \{\alpha \cdot 1 : \alpha \in \mathbb{C}\}, \quad \mathcal{H}_0 = \{\alpha \cdot 1 : \alpha \in \mathbb{C}\} \end{aligned}$$

We claim that e is an inner product on E. To see this, first note that for any ϕ, $\psi \in \mathcal{D}(A)$, we have

$$b(B^{-1}A\phi, \psi) = (A\phi, \psi) = (\phi, A\psi) = b(B^{-1}\phi, A\psi) = b(\phi, B^{-1}A\psi).$$

Thus $B^{-1}A$ is self-adjoint relative to $b(\cdot\,,\cdot)$. Let $(\lambda_k)_{k=0}^{\infty}$ denote the eigenvalues of $B^{-1}A$ and let (ϕ_k) denote the corresponding eigenfunctions, which are normalized so that $b(\phi_k, \phi_k) = 1$, all k. Since $b(B^{-1}A\phi, \phi) = a(\phi, \phi)$, the eigenvalues are non-negative:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots$$

and since $(B^{-1}A + \lambda I)^{-1}$ is compact for $\lambda > 0$ (by 2.16), $\lambda_k \to \infty$ as $k \to \infty$. Note that

$$B^{-1}A\phi_k = \lambda_k \phi_k \quad \Leftrightarrow \quad A\phi_k = \lambda_k B\phi_k.$$

We note also that ϕ_0 is a constant function, and by the fact that $B^{-1}A$ is self-adjoint, all other eigenfunctions are orthogonal to ϕ_0 relative to $b(\cdot,\cdot)$. Thus we have the orthogonal (with respect to b) decomposition:

$$\mathcal{V} \times \mathcal{H} = E \oplus Z,$$

where both E and Z are $\mathcal{B}^{-1}\mathcal{A}$-invariant spaces.

If $u \in \tilde{\mathcal{V}}$ then since λ_1 is positive and the other eigenvalues are no smaller we have

$$a(u,u) \geq a(u,u)/2 + \lambda_1 b(u,u)/2 \geq C\|u\|_{\mathcal{V}}^2,$$

where $C > 0$ by (2.16). It thus immediately follows that there exists $C' > 0$ such that

$$e(\{u,v\},\{u,v\}) \geq C'(\|u\|_{\mathcal{V}}^2 + \|v\|_{\mathcal{H}}^2) \quad \forall \{u,v\} \in \tilde{\mathcal{V}} \times \tilde{\mathcal{H}}.$$

Hence $e(\cdot, \,, \cdot)$ is an inner product on E and consequently by Stone's theorem [11], $\mathcal{B}^{-1}\mathcal{A}$ is the generator of a unitary group on E.

The dynamics on Z are those that describe rigid motions. Suppose $\{w_0, v_0\} \in Z$. Then $\mathcal{B}^{-1}\mathcal{A}\{w_0, v_0\} = \{v_0, 0\}$. It follows that

$$e^{\mathcal{B}^{-1}\mathcal{A}t}\{w_0, v_0\} = \{w_0 + tv_0, v_0\}. \tag{2.19}$$

We summarize these facts in the following proposition.

Proposition 2.3. *$\mathcal{B}^{-1}\mathcal{A}$ is the infinitesimal generator of a strongly continuous group $\mathbb{T} = (\mathbb{T}_t)_{t\in\mathbb{R}}$ on $\mathcal{V}\times\mathcal{H} = E\oplus Z$. The restriction of $\mathbb{T}$ to the invariant subspace E is unitary and the restriction of $\mathbb{T}$ to the invariant, two dimensional subspace Z is given by (2.19). Consequently, given the initial conditions $\{w, w_t\}_{t=0} = \{w^0, w^1\}$ in $\mathcal{V} \times \mathcal{H}$ there exists a unique solution w to (2.12), (2.13) with*

$$w \in C([0,\infty); \mathcal{V}) \cap C^1([0,\infty); \mathcal{H}).$$

Furthermore, for each $t > 0$

$$a(w(t), w(t)) + b(w_t(t), w_t(t)) = a(w(0), w(0)) + b(w_t(0), w_t(0)).$$

The semigroup $\mathbb{T}$ in Proposition 2.3 may also be extended by duality to an isomorphic semigroup on the space $\mathcal{H} \times \mathcal{V}'$. To see this, we set $X = X' = \mathcal{V} \times \mathcal{H}$ and $X_1 = \mathcal{D}(\mathcal{A}) \times \mathcal{V}$. Then $X_1' = \mathcal{H} \times \mathcal{V}'$. Also, we have that $X_1 = E_1 \oplus Z$, where $E_1 = (\mathcal{D}(A) \cap \tilde{\mathcal{V}}) \times \tilde{\mathcal{V}}$. Thus $X_1' = E_1' \oplus Z$, where E_1' is dual to E_1 with respect to the inner product e (and pivot space E). Thus we keep the same definition of $\mathbb{T}$ on Z. Since $i\mathcal{B}^{-1}\mathcal{A}$ is self-adjoint and continuous from E_1 onto E, it has a self-adjoint extension that is an isomorphism from E to E_1'. Therefore we define $\mathbb{T}$ on E_1' by duality, as follows:

$$e(\mathbb{T}y, Y) \stackrel{\text{def}}{=} e(\mathbb{T}(i\mathcal{B}^{-1}\mathcal{A})^{-1}y\,, (i\mathcal{B}^{-1}\mathcal{A})Y), \quad \forall\, y \in E_1', \forall\, Y \in E_1.$$

Thus we have the following.

Corollary 2.4. *The semigroup $\mathbb{T}$ in Proposition 2.3 has a continuous extension to a strongly continuous semigroup on $\mathcal{H} \times \mathcal{V}'$. Thus given the initial conditions $\{w, w_t\}_{t=0} = \{w^0, w^1\}$ in $\mathcal{H} \times \mathcal{V}'$ there exists a unique solution w to (2.12), (2.13) with*

$$w \in C([0,\infty); \mathcal{H}) \cap C^1([0,\infty); \mathcal{V}').$$

Remark 2.2. Proposition 2.3 and its corollary remain true under any of the usual types of boundary conditions one considers for the Euler-Bernoulli beam (e.g., clamped, simply supported, etc.). Thus the boundary conditions (2.13) are not essential for obtaining existence and uniqueness of the coupled system.

3. Main estimates.

In this section we prove the main estimates relating to the eigenvalues and eigenfunctions of $B^{-1}A$ that will be needed for the proof of Theorem 1.1.

Recall that $(\lambda_k)_{k=0}^{\infty}$ denotes the sequence of eigenvalues of $B^{-1}A$, arranged in increasing order and (ϕ_k) denotes the corresponding sequence of eigenfunctions, normalized so that $b(\phi_k, \phi_k) = 1$, for all k. We let (λ_k^0) denote the eigenvalues of A, also arranged in increasing order and (e_k) denote the eigenfunctions of A, normalized so that $(e_k, e_k) = 1$, all k. Explicitly,

$$\lambda_k^0 = EIk^4\pi^4/m, \qquad k = 0, 1, 2, \dots\,,$$

while

$$e_k(x) = \sqrt{2}\cos k\pi x,\ \ k = 1, 2, \dots\,; \quad e_0 = 1.$$

The following lemma contains the key perturbation estimate that is needed to prove Theorem 1.1. Recall that $\mu = \rho/m$.

Lemma 3.1. *For any $\mu \geq 0$ and $k = 1, 2, \dots$ the following holds:*

$$\left(1 + \mu\frac{c_0}{k}\right)^{-1} \lambda_k^0 \leq \lambda_k \leq \lambda_k^0, \tag{3.1}$$

where $c_0 = \frac{1}{\pi}\|\Lambda_0\|_{\mathcal{L}(\mathcal{H}, H^1(0,1))}$.

Proof. Since $B^{-1}A$ is self-adjoint relative to $b(\cdot\,,\cdot)$ it follows from the minimax property of the eigenvalues that for $k = 1, 2, \dots$

$$\lambda_k = \max_{\dim H = k}\ \min_{u \in H^\perp}\ \frac{b(B^{-1}Au, u)}{b(u,u)} = \max_{\dim H = k}\ \min_{u \in H^\perp}\ \frac{(Au, u)}{\|u\|^2} \cdot \frac{\|u\|^2}{(Bu, u)},$$

where H denotes a subspace of $\mathcal{H}$ and $H^{\perp} = \{u \in \mathcal{H} \mid b(u,v) = 0,\ \forall v \in H\}$. Let

$$H_k = \operatorname{span}\{B^{-1}e_0, B^{-1}e_1, \dots, B^{-1}e_{k-1}\}.$$

Then $H_k^{\perp} = $ closed span$\{e_k, e_{k+1}, \dots\}$ and we have

$$\lambda_k \geq \min_{u \in H_k^{\perp}} \frac{(Au,u)}{\|u\|^2} \cdot \frac{\|u\|^2}{(Bu,u)} \geq \left(\min_{u \in H_k^{\perp}} \frac{(Au,u)}{\|u\|^2}\right)\left(\max_{u \in H_k^{\perp}} \frac{(Bu,u)}{\|u\|^2}\right)^{-1}$$
$$= \lambda_k^0 \left(1 + \mu \max_{u \in H_k^{\perp}} \frac{(\Lambda_0 u, u)}{\|u\|^2}\right)^{-1}.$$

If $u \in H_k^{\perp}$ then $u = \sum_{j=k}^{\infty} u_j e_j$, with $(u_j) \in l^2$. Since Λ_0 is continuous from $\mathcal{H} \to H^1(0,1)$ it follows that $\frac{d}{dx}\Lambda_0 u$ may be expanded in a sine series which converges in $L^2(0,1) = \mathcal{H}$. Thus for any $N > k$

$$\sum_{j=k}^{N} j^2 |(\Lambda_0 u, e_j)|^2 = \sum_{j=k}^{N} j^2 \left|\left(\frac{d}{dx}\Lambda_0 u, \frac{\sqrt{2}\sin j\pi x}{j\pi}\right)\right|^2$$
$$\leq \pi^{-2}\|\Lambda_0 u\|^2_{H^1(0,1)} \leq c_0^2 \|u\|^2.$$

Therefore

$$|(\Lambda_0 u, u)| = |\sum_{j=k}^{\infty} u_j (\Lambda_0 u, e_j)| \leq \sum_{j=k}^{\infty} |u_j/j||j(\Lambda_0 u, e_j)|$$
$$\leq \left(\sum_{j=k}^{\infty} |u_j/j|^2\right)^{1/2} c_0\|u\| \leq \frac{c_0\|u\|^2}{k}.$$

Thus $\lambda_k \geq \lambda_k^0(1 + \mu c_0/k)^{-1}$. Furthermore, due to the fact that $(Bu,u)/\|u\|^2 > 1$, it easily follows from the minimax property that $\lambda_k \leq \lambda_k^0$ for all k. $\square$

Lemma 3.2. *Let $l \in \mathbb{N}$. If $\mu \in [0, 4l/c_0)$ then there exists $\delta > 0$ such that*

$$\sqrt{\lambda_{k+l}} - \sqrt{\lambda_k} > \delta k, \qquad \forall k \in \mathbb{N}. \tag{3.2}$$

Proof. For appropriate C^0, we have $\lambda_k^0 = C^0 k^4$, $k = 0, 1, \dots$. By Lemma 3.1

$$\lambda_{k+l} - \lambda_k \geq \frac{\lambda_{k+l}^0}{1 + \mu c_0/(k+l)} - \lambda_k^0 = C^0\left(\frac{(k+l)^4}{1 + \mu c_0/(k+l)} - k^4\right) > C^0 \frac{4l - \mu c_0}{1 + \mu c_0} k^3.$$

Also by Lemma 3.1, we have $\lambda_k \leq \lambda_k^0 = C^0 k^4$. Therefore we have

$$\sqrt{\lambda_{k+l}} - \sqrt{\lambda_k} \geq C^0 \frac{4l - \mu c_0}{1 + \mu c_0} \frac{k^3}{\sqrt{\lambda_{k+l}} + \sqrt{\lambda_k}} \geq \frac{\sqrt{C^0}}{2(l+1)^2} \frac{4l - \mu c_0}{1 + \mu c_0} k.$$

Thus the required estimates hold. $\square$

Lemma 3.3. *Let $l \in \mathbb{N}$. If $\mu \in [0, 4l/c_0)$ we have*

$$\sum_{j:j\neq k}^{\infty} |(e_j, \phi_k)| \leq C(\mu, l)\mu + \sqrt{(l-1)}, \qquad k = 1, 2, \ldots \tag{3.3}$$

where

$$C(\mu, l) = \left(\pi + \frac{\pi}{8\sqrt{2}} + \frac{\pi}{\sqrt{2}}(l+1)^4 + (l+1)^4 \frac{1+\mu c_0}{4l - \mu c_0}\right) c_0.$$

Proof. Since $1 = b(\phi_k, \phi_k) = \|\phi_k\|^2 + \mu(\Lambda_0 \phi_k, \phi_k)$ and $c_0 = \frac{1}{\pi}\|\Lambda_0\|_{\mathcal{L}(\mathcal{H}, H^1)}$ we have

$$1/(1 + \pi\mu c_0) \leq \|\phi_k\|^2 \leq 1, \qquad k = 0, 1, \ldots. \tag{3.4}$$

Recall that for all k, $A\phi_k = \lambda_k B\phi_k = \lambda_k(I + \mu\Lambda_0)\phi_k$ and $Ae_k = \lambda_k^0 e_k = C^0 k^4 e_k$. Therefore for any $k, j = 0, 1, \ldots$

$$\lambda_j^0(\phi_k, e_j) = \lambda_k(\phi_k, e_j) + \lambda_k \mu(\Lambda_0 \phi_k, e_j)$$

which implies

$$(\phi_k, e_j) = \mu \frac{\lambda_k}{\lambda_j^0 - \lambda_k}(\Lambda_0 \phi_k, e_j), \qquad \text{if } \lambda_j^0 \neq \lambda_k. \tag{3.5}$$

We have

$$\sum_{j:j\neq k}^{\infty} |(\phi_k, e_j)| = \sum_{j=0}^{k-1} |(\phi_k, e_j)| + \sum_{j=k+1}^{\infty} |(\phi_k, e_j)|.$$

Using (3.1), (3.4), (3.5) we obtain

$$\sum_{j=k+1}^{\infty} |(\phi_k, e_j)| = \mu \sum_{j=k+1}^{\infty} \frac{\lambda_k}{\lambda_j^0 - \lambda_k} |(\Lambda_0 \phi_k, e_j)|$$

$$\leq \mu \sum_{j=k+1}^{\infty} \frac{\lambda_k^0}{\lambda_j^0 - \lambda_k^0} |(\Lambda_0 \phi_k, e_j)| = \mu \sum_{j=k+1}^{\infty} \frac{k^4}{j^4 - k^4} |(\Lambda_0 \phi_k, e_j)|$$

$$= \mu \sum_{j=k+1}^{\infty} \frac{k^4}{j^4 - k^4} \left| \left(\frac{d}{dx}\Lambda_0 \phi_k, \frac{\sqrt{2}\sin j\pi x}{j\pi} \right) \right| \leq \mu \left(\sum_{j=k+1}^{\infty} \frac{k^8}{(j^4 - k^4)^2 j^2} \right)^{1/2} \frac{\|\Lambda_0 \phi_k\|_{H^1}}{\pi}$$

$$\leq \mu \left(\sum_{j=k+1}^{\infty} \frac{1}{(j-k)^2} \right)^{1/2} \frac{1}{4} c_0 \|\phi_k\| \leq \frac{\pi}{8\sqrt{2}} c_0 \mu.$$

To estimate $\sum_{j=0}^{k-1} |(\phi_k, e_j)|$ we consider two cases: $k > l$ and $k \leq l$.

If $k > l$ then we can write $\sum_{j=0}^{k-1} |(\phi_k, e_j)|$ in the form

$$\sum_{j=0}^{k-1} |(\phi_k, e_j)| = |(\phi_k, e_0)| + |(\phi_k, e_{k-l})| + \sum_{j=1}^{k-l-1} |(\phi_k, e_j)| + \sum_{j=k-l+1}^{k-1} |(\phi_k, e_j)|.$$

From (3.1) it follows that for any $k > l$

$$\lambda_k - \lambda^0_{k-l} \geq \frac{\lambda^0_k}{1+\mu c_0/k} - \lambda^0_{k-l} = C^0 \frac{k^4 - (k-l)^4(1+\mu c_0/k)}{1+\mu c_0/k} \geq C^0 \frac{(4l - \mu c_0)(k-l)^3}{1+\mu c_0/k}.$$

Therefore for any $\mu \in (0, 4l/c_0)$ and any $k > l$ we have

$$|(\phi_k, e_0)| = \mu |(\Lambda_0 \phi_k, 1)| \leq \mu \|\Lambda_0\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \leq \mu \|\Lambda_0\|_{\mathcal{L}(\mathcal{H},H^1)} = \pi c_0 \mu,$$

$$\begin{aligned} |(\phi_k, e_{k-l})| &= \mu \frac{\lambda_k}{\lambda_k - \lambda^0_{k-l}} |(\Lambda_0 \phi_k, e_{k-l})| \leq \mu \frac{k^4(1+\mu c_0/k)}{(4l-\mu c_0)(k-l)^3} |(\Lambda_0 \phi_k, e_{k-l})| \\ &\leq \mu \frac{1+\mu c_0}{4l - \mu c_0} \frac{k^4}{(k-l)^3} \left| \left(\frac{d}{dx} \Lambda_0 \phi_k, \frac{\sqrt{2}\sin(k-l)\pi x}{(k-l)\pi} \right) \right| \leq (l+1)^4 \frac{1+\mu c_0}{4l-\mu c_0} c_0 \mu \end{aligned}$$

and

$$\sum_{j=k-l+1}^{k-1} |(\phi_k, e_j)| \leq \sqrt{l-1} \left(\sum_{j=k-l+1}^{k-1} |(\phi_k, e_j)|^2 \right)^{1/2} \leq \sqrt{l-1}.$$

Before estimating $\sum_{j=1}^{k-l-1} |(\phi_k, e_j)|$ we note that for any $k > l$

$$\begin{aligned} \sum_{j=1}^{k-l-1} \frac{k^8}{((k-l)^4 - j^4)^2 j^2} &= \sum_{j=1}^{k-l-1} \frac{k^8}{(k-l-j)^2(k-l+j)^2((k-l)^2+j^2)^2 j^2} \\ &\leq \frac{k^8}{(k-l)^6} \sum_{j=1}^{k-l-1} \frac{1}{(k-l-j)^2 j^2} = \frac{k^8}{(k-l)^8} \sum_{j=1}^{k-l-1} \left(\frac{1}{k-l-j} + \frac{1}{j} \right)^2 \\ &\leq 4(l+1)^8 \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{2}(l+1)^8. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{j=1}^{k-l-1} |(\phi_k, e_j)| &= \mu \sum_{j=1}^{k-l-1} \frac{\lambda_k}{\lambda_k - \lambda^0_j} |(\Lambda_0 \phi_k, e_j)| \leq \mu \sum_{j=1}^{k-l-1} \frac{\lambda^0_k}{\lambda^0_{k-l} - \lambda^0_j} |(\Lambda_0 \phi_k, e_j)| \\ &\leq \mu \sum_{j=1}^{k-l-1} \frac{k^4}{(k-l)^4 - j^4} \left| \left(\frac{d}{dx} \Lambda_0 \phi_k, \frac{\sqrt{2}\sin j\pi x}{j\pi} \right) \right| \\ &\leq \mu \left(\sum_{j=1}^{k-l-1} \frac{k^8}{((k-l)^4 - j^4)^2 j^2} \right)^{1/2} \frac{\|\Lambda_0 \phi_k\|_{H^1}}{\pi} \leq \frac{\pi}{\sqrt{2}} (l+1)^4 c_0 \mu. \end{aligned}$$

Thus in the case $k > l$ we have

$$\sum_{j=0}^{k-1} |(\phi_k, e_j)| \leq \left(\pi + \frac{\pi}{\sqrt{2}} (l+1)^4 + (l+1)^4 \frac{1+\mu c_0}{4l - \mu c_0} \right) c_0 \mu + \sqrt{l-1}.$$

In the case $k \le l$ we obtain

$$\sum_{j=0}^{k-1} |(\phi_k, e_j)| = |(\phi_k, e_0)| + \sum_{j=1}^{l-1} |(\phi_k, e_j)| \le \pi c_0 \mu + \sqrt{l-1}.$$

Inequality (3.3) follows from the above estimates. $\square$

Define the sequence $(b_k)_{k=0}^{\infty}$ by

$$b_k = \langle \delta_0, \phi_k \rangle = \phi_k(0), \quad k = 0, 1, \dots . \tag{3.6}$$

In the above, δ_0 denotes the Dirac delta function with mass at $x = 0$. Note that the sequence (b_k) is well-defined since $\delta_0 \in \mathcal{V}'$.

Lemma 3.4. *Let (b_k) be the sequence defined in (3.6). Then:*

(1) *$(b_k) \in l^\infty$, i.e., (b_k) is a bounded sequence.*

(2) *For any $\epsilon \in (0, 1)$, $k = 0, 1, \dots$ we have $|b_k| > \epsilon$ for any $\mu \in (0, \mu_0(\epsilon))$ where $\mu_0(\epsilon) \in (0, 4/c_0)$ is uniquely determined as the solution of the following implicit equation*

$$F(\mu_0) = \frac{1}{1 + \pi \mu_0 c_0} - C^2(\mu_0, 1)\mu_0^2 - \sqrt{2} C(\mu_0, 1)\mu_0 = \epsilon.$$

Proof. (1) Since

$$\delta_0 = e_0 + \sum_{j=1}^{\infty} \sqrt{2} e_j,$$

where the sum converges in $\mathcal{V}'$, we can write

$$|b_k| = \left| (e_0, \phi_k) + \sqrt{2} \sum_{j=1}^{\infty} (e_j, \phi_k) \right|, \qquad \forall k \in \mathbb{N}. \tag{3.7}$$

Let l be such that $\mu \in (0, 4l/c_0)$. Using (3.3), (3.4) we obtain

$$|b_k| \le \left| \sqrt{2} \sum_{j=0}^{\infty} (e_j, \phi_k) \right| \le \sqrt{2} |(e_k, \phi_k)| + \sqrt{2} \sum_{j:j\ne k}^{\infty} |(e_j, \phi_k)|$$

$$\le \sqrt{2} \left(1 + C(\mu, l)\mu + \sqrt{l-1} \right).$$

(2) From (3.3), (3.4), (3.7) it follows that for any $\mu \in (0, 4/c_0)$

$$\|\phi_k\|^2 - |(\phi_k, e_k)|^2 = \sum_{j:j\ne k}^{\infty} |(\phi_k, e_j)|^2 \le C^2(\mu, 1)\mu^2,$$

$$|(\phi_k, e_k)| \ge |(\phi_k, e_k)|^2 \ge \left(\|\phi_k\|^2 - C^2(\mu, 1)\mu^2 \right) \ge \frac{1}{1 + \pi \mu c_0} - C^2(\mu, 1)\mu^2,$$

$$|b_k| \ge |(e_k, \phi_k)| - \sqrt{2} \sum_{j:j\ne k}^{\infty} |(e_j, \phi_k)| \ge \frac{1}{1 + \pi \mu c_0} - C^2(\mu, 1)\mu^2 - \sqrt{2} C(\mu, 1)\mu = F(\mu).$$

It is easy to see that $F(\mu)$ is a continuous decreasing function on $[0, 4/c_0)$ that satisfies $F(0) = 1$ and $\lim_{\mu \to 4/c_0} F(\mu) = -\infty$. Therefore for any $\epsilon \in (0, 1)$ there exists a unique $\mu_0 = \mu_0(\epsilon) \in (0, 4/c_0)$ such that $F(\mu_0) = \epsilon$ and $F(\mu) > \epsilon$ for $0 < \mu < \mu_0$. The proof of Lemma 3.4 is complete. $\square$

4. Proof of main results.

The boundary value problem (1.1)–(1.7) can be rewritten

$$
\begin{aligned}
mw_{tt} + \rho\Lambda_0 w_{tt} + EIw_{xxxx} &= 0 \qquad \text{on } (0,1)\times\mathbb{R}^+, && (4.1)\\
w_x(0,t) = w_x(1,t) = w_{xxx}(1,t) &= 0; \\
EIw_{xxx}(0,t) &= f(t) \qquad \text{on } \mathbb{R}^+, && (4.2)\\
w(x,0) = w^0(x), \qquad w_t(x,0) &= w^1(x) \qquad \text{on } (0,1), && (4.3)
\end{aligned}
$$

where $\{w^0, w^1\}$ are given in $\mathcal{V}\times\mathcal{H}$ and $f \in L^2(0,\infty)$.

An integration by parts of (4.1) against an element of $\mathcal{V}$ gives the following variational formulation of (4.1)–(4.3): Find w satisfying

$$
\begin{aligned}
& w \in C([0,\infty);\mathcal{V}) \cap C^1([0,\infty);\mathcal{H}), && (4.4)\\
& \frac{d}{dt} b(w_t,\hat{w}) + a(w,\hat{w}) = m^{-1}\hat{w}(0)f(t), \qquad \forall\, \hat{w}\in\mathcal{V}, && (4.5)\\
& \{w, w_t\}|_{t=0} = \{w^0, w^1\}, && (4.6)
\end{aligned}
$$

where (4.5) holds in the sense of distributions on $[0,\infty)$.

Before we prove Theorem 1.1 we need to briefly discuss Riesz bases.

A sequence $(p_k)_{k=1}^{\infty}$ is a *Riesz basis* for the Hilbert space X if it is the image under a Hilbert space isomorphism $F : X \to X$ of an orthonormal basis. Corresponding to the Riesz basis (p_k) is a uniquely defined *biorthogonal sequence* $(q_k) \in X$ which satisfies

$$
(p_k, q_j)_X = \delta_{kj} = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j \end{cases}.
$$

The sequence (q_k) is itself a Riesz basis for X. If $f \in X$ then there is a unique sequence $(f_k) \in l^2$ such that $f = \sum_{k=1}^{\infty} f_k p_k$ and there exists positive constants c, C such that

$$
c\|(f_k)\|_{l^2} \leq \|f\|_X \leq C\|(f_k)\|_{l^2}. \qquad (4.7)
$$

In addition,

$$
f_k = (f, q_k)_X, \quad \forall k \in \mathbb{N}.
$$

We refer the reader to Young [16] for details.

Proof of Theorem 1.1. The corresponding first order form of (4.4)–(4.6) analogous to (2.18) is given by

$$
\frac{d}{dt} W = \mathcal{B}^{-1}\mathcal{A}W + \mathcal{B}^{-1}\begin{pmatrix} 0 \\ m^{-1}\delta_0 \end{pmatrix} f(t), \qquad (4.8)
$$

with initial conditions

$$
W(0) = W^0 = \begin{pmatrix} w^0 \\ w^1 \end{pmatrix},
$$

where δ_0 denotes the element of $\mathcal{V}'$ for which $\langle \delta_0, u\rangle = u(0)$, for all $u \in \mathcal{V}$.

Due to the fact that the nonhomogeneous term in (4.8) belongs to $L^1_{\mathrm{loc}}(\mathcal{H}\times\mathcal{V}')$ and the semigroup $\mathbb{T}$ is well-posed on $\mathcal{H}\times\mathcal{V}'$ (by Corollary 4.4), a mild solution (with values in $\mathcal{H}\times\mathcal{V}'$) is given by the variation of constants formula:

$$W(t) = \mathbb{T}_t W^0 + \int_0^t \mathbb{T}_{t-s} \begin{pmatrix} 0 \\ m^{-1}B^{-1}\delta_0 \end{pmatrix} f(s)\,ds \tag{4.9}$$

The standard semigroup theory gives $W \in C([0,T];\mathcal{H}\times\mathcal{V}')$, for any $T>0$. However this is suboptimal and our first goal is to show that (4.4) holds, i.e., $W \in C([0,T];\mathcal{V}\times\mathcal{H})$.

Since $W^0 \in \mathcal{V}\times\mathcal{H}$, by Proposition 2.3 we know that $\mathbb{T}_t W^0 \in C([0,\infty);\mathcal{V}\times\mathcal{H})$. Thus, for purposes of proving that (4.4) holds, we may assume without loss of generality that

$$W^0 = 0. \tag{4.10}$$

Recall that $B^{-1}A$ has eigenvalues (λ_k) for $k=0,1,2,\dots$ and corresponding eigenfunctions (ϕ_k) which are normalized so that $b(\phi_k,\phi_k)=1$, for all k. Corresponding to the eigenvalue λ_k of $B^{-1}A$ there are two eigenvalues $\pm i\sqrt{\lambda_k}$ of $\mathcal{B}^{-1}\mathcal{A}$ for $k=1,2,\dots$ with corresponding eigenvectors

$$\Phi_k^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_k/\sqrt{\lambda_k} \\ i\phi_k \end{pmatrix}, \qquad \Phi_k^- = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_k/\sqrt{\lambda_k} \\ -i\phi_k \end{pmatrix}$$

which satisfy

$$\mathcal{A}\Phi_k^\pm = \pm i\sqrt{\lambda_k}\mathcal{B}\Phi_k^\pm.$$

Corresponding to $\lambda_0=0$, $\mathcal{B}^{-1}\mathcal{A}$ has the eigenfunction and generalized eigenfunction

$$\Phi_0^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Phi_0^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

These satisfy: $\mathcal{B}^{-1}\mathcal{A}\Phi_0^+ = 0$; $\mathcal{B}^{-1}\mathcal{A}\Phi_0^- = \Phi_0^+$. The eigenfunctions $\Phi_k^\pm$, $k=1,2,\dots$ form a basis for the space E and are orthonormal relative to the inner product e and the eigenfunctions $\Phi_0^\pm$ are an orthogonal basis for the space Z.

Let $t>0$. By calculating $e(W(t),\Phi_k^\pm)$, $k=1,2,\dots$ on both sides of the equation in (4.9) and using (4.10) we obtain

$$W_k^\pm = 2^{-1/2}\int_0^t e^{\pm i\sqrt{\lambda_k}(t-s)}(\mp i m^{-1} b_k) f(s)\,ds, \quad k\in\mathbb{N}, \tag{4.11}$$

where $W_k^\pm = e(W(t),\Phi_k^\pm)$ and the sequence (b_k) is defined by (3.6). In calculating (4.11) we have used: $e(\mathcal{B}^{-1}\{0, m^{-1}\delta_0\},\Phi_k^\pm) = b(B^{-1}m^{-1}\delta_0, \pm i\phi_k/\sqrt{2}) = \mp i m^{-1}\langle \delta,\phi_k\rangle/\sqrt{2}$. For the case $k=0$ we calculate the projection onto Z of both sides of (4.9) then calculate, using (2.19), the components in terms of $\Phi_0^\pm$. We obtain

$$W_0^+ = \int_0^t (t-s)m^{-1}b_0 f(s)\,ds \tag{4.12}$$

$$W_0^- = \int_0^t m^{-1}b_0 f(s)\,ds \tag{4.13}$$

where, letting $W(t) = \{w_1, w_2\}$, $W_0^+ = b(w_1, 1)$, $W_0^- = b(w_2, 1)$.

By letting $\tilde{f}(\tau) = m^{-1} f(t-\tau)$, $p_k^{\pm}(\tau) = e^{\pm i\sqrt{\lambda_k}\tau}$ for $k = 1, 2, \dots$, $p_0^+(\tau) = \tau$, $p_0^-(\tau) = 1$, we can rewrite (4.11)–(4.13), more compactly as

$$\gamma_k^{\pm} = \int_0^t b_k p_k^{\pm}(s) \tilde{f}(s)\, ds, \quad k = 0, 1, 2, \dots, \tag{4.14}$$

where $\gamma_k^{\pm} = \pm i\sqrt{2} W_k^{\pm}$ for all $k \in \mathbb{N}$ and $\gamma_0^{\pm} = W_0^{\pm}$.

Let l be such that $\rho < 4ml/c_0$, where c_0 is defined in Lemma 3.1. Then by Lemma 3.2, $\sqrt{\lambda_{k+l}} - \sqrt{\lambda_k}$ tends to infinity as $k \to \infty$. It follows that the sequence $(\sqrt{\lambda_k})$ can be written as a union of l sequences:

$$(\sqrt{\lambda_k}) = (\sigma_k^1)_{k=1}^{\infty} \cup \ldots \cup (\sigma_k^l)_{k=1}^{\infty}, \tag{4.15}$$

where each of the sequences (σ_k^j), $j = 1, \dots, l$ satisfies

$$\sigma_{k+1}^j - \sigma_k^j > \epsilon_j k, \quad \epsilon_j > 0. \tag{4.16}$$

Let us first examine the case $l = 1$.

If $l = 1$ it follows from the separation condition (4.16) that the sequence $(p_k^{\pm})$ forms a Riesz basis for the space $M \subset L^2(0,t)$ defined by

$$M = \text{closed span}\,(\{p_k^{\pm}\}_{k=0}^{\infty}),$$

(see Young, [16, pp. 162-166]; actually this result applies to $(p_k^{\pm})_{k=1}^{\infty}$, however it is easy to show that independence is preserved when p_0^+ and p_0^- are included.) By Lemma 3.4 we know that $(b_k) \in l^{\infty}$ and consequently by (4.7) we have for some $K_1 > 0$

$$\|(W_k^{\pm})\|_{l^2} \le \|(\gamma_k^{\pm})\|_{l^2} \le K_1 \|\tilde{f}\|_{L^2(0,t)}. \tag{4.17}$$

Since $(\Phi_k^{\pm})$ forms an orthonormal basis for $E \oplus Z$ it follows that $\|W(t)\|_{E\oplus Z} = \|(W_k^{\pm})\|_{l^2}$. Thus, since $\mathcal{V} \times \mathcal{H} = E \oplus Z$ (with equivalent topologies) from (4.17) there exists $\tilde{K}_1 > 0$ such that

$$\|W(t)\|_{\mathcal{V}\times\mathcal{H}} \le \tilde{K}_1 \|\tilde{f}\|_{L^2(0,t)}.$$

From this it follows (for example, see Weiss, [15]) that for any $T > 0$

$$W \in C([0,T]; \mathcal{V} \times \mathcal{H}), \qquad \forall\, f \in L^2(0,T)$$

and

$$\|W\|_{L^{\infty}((0,T);\mathcal{V}\times\mathcal{H})} \le C\|f\|_{L^2(0,T)},$$

for some $C = C(T) > 0$. Thus (4.4) holds.

If $l > 1$ we apply the same argument; in place of (4.17) we obtain

$$\|(W_k^{\pm})\|_{l^2} \le (K_1 + \ldots + K_l)\|\tilde{f}\|_{L^2(0,t)},$$

for appropriate constants $K_1, \dots, K_l$ and again obtain the regularity in (4.4).

Finally, using that (4.8) holds on $\mathcal{H} \times \mathcal{V}'$ and (4.9) holds on $\mathcal{V} \times \mathcal{H}$ it is easy to verify that (4.5) and (4.6) are satisfied. The uniqueness follows from the uniqueness of the homogeneous problem.

Now we consider the control problem. Suppose we wish to find a control which drives the initial state $W^0 \in \mathcal{V} \times \mathcal{H}$ to the terminal state $W^T \in \mathcal{V} \times \mathcal{H}$ in time T. Since (4.9) is valid on $\mathcal{V} \times \mathcal{H}$, the problem is the same as finding a control which drives the initial state 0 to the terminal state $W^T - \mathbb{T}_T W^0$. Therefore it is enough to assume that $W^0 = 0$ find $f \in L^2(0,T)$ for which $W(T) = W^T$.

Corresponding to W^T is a uniquely defined sequence $(\gamma_k^{\pm}) \in l^2$ with $\|(\gamma_k^{\pm})\|_{l^2} \leq C\|W^T\|_{\mathcal{V}\times\mathcal{H}}$, for some $C > 0$. We wish to find $\tilde{f} \in L^2(0,T)$ that solves the moment problem (4.14) (with $t = T$). By Lemma 3.4 there exists $\mu_0 > 0$ such that if $\rho < m\mu_0$ then $(|b_k|^{-1})_{k=0}^{\infty} \in l^{\infty}$. Thus, for such ρ, we have $(\gamma_k^{\pm}/b_k) \in l^2$. By Lemma 3.2 the eigenvalue separation condition holds. Thus $(p_k^{\pm})$ forms a Riesz basis for M, its closed span in $L^2(0,T)$. Let $(q_k^{\pm})$ denote the biorthogonal sequence to $(p_k^{\pm})$ in M. We let

$$\tilde{f} = \sum_{k=0}^{\infty}(\gamma_k^+/b_k)q_k^+ + \sum_{k=0}^{\infty}(\gamma_k^-/b_k)q_k^- . \tag{4.18}$$

It follows from (4.7) that the series for $\tilde{f}$ converges in $L^2(0,T)$ and it is easy to see from the biorthogonality that $\tilde{f}$ indeed solves the moment problem (4.14). This concludes the proof of Theorem 1.1. $\square$

Remark 4.3. Due to the fact that the eigenvalue separation tends to ∞ as $k \to \infty$ (see Lemma 3.2 with $l = 1$) the space M in the proof of Theorem 1.1 is a proper subspace of $L^2(0,T)$, for any $T > 0$. Consequently, if f is the control constructed in the above proof that drives $W^0 = 0$ to W^T in time T then $f(t) + g(T-t)$, for any $g \in M^{\perp}$, also drives W^0 to W^T in time T. The control constructed in the proof of Theorem 1.1 is the unique control of minimal norm in $L^2(0,T)$ which drives the initial state $W^0 = 0$ to the terminal state.

5. Extensions and related results.

The methods used in this paper can be applied to several similar situations involving other types of boundary conditions and different dynamics on the flexible boundary. We conclude this paper with a brief discussion of the results one can obtain in some of these situations.

5.1. Other boundary conditions for the beam. Concerning the *well-posedness* of (1.1)–(1.7), any of the usual types of boundary conditions that are associated with the Euler-Bernoulli beam may used in place of those in (1.6). This was already mentioned in Remark 2.2.

Concerning the *exact controllability* however, our approach is valid only for boundary conditions. In Lemma 3.1 we needed to use the estimate

$$u = \sum_{j=k}^{m} u_j \cos j\pi x \Rightarrow (\Lambda_0 u, u)_{L^2(0,1)} \leq \frac{c}{k}\|u\|^2_{L^2(0,1)}, \tag{5.1}$$

where c is independent of k and m. In other words, the Neumann to Dirichlet map, which we know is continuous from $L^2(0,1)$ to $H^1(0,1)$, is shown in (5.1) to also produce exactly one degree of smoothing in terms of Fourier series. However, in the case of simply supported boundary conditions:

$$w(0,t) = w(1,t) = w_{xx}(0,t) = 0;\ w_{xx}(1,t) = f(t)$$

then we would have to prove (5.1) with sines in place of the cosines. Unfortunately, unless we know that Λ_0 maps into H_0^1 we can not conclude that an estimate like (5.1) holds. Therefore our approach fails to prove controllability for types of boundary conditions involving *fixed end conditions.*

On the other hand, we can consider control of boundary conditions such as:

$$w_{xx}(0,t) = w_{xx}(1,t) = w_{xxx}(0,t) = 0;\ w_{xxx}(1,t) = f(t). \tag{5.2}$$

In this case the eigenfunctions of A satisfy all the properties that were required of the cosines, namely orthogonality, the same growth rate of eigenvalues and the property (5.1). Consequently a result like Theorem 1.1 is valid when the boundary conditions (4.2) are replaced by those in (5.2). (There is one difference: with (5.2) we get exact controllability in a quotient space, where the the zero element of this quotient space is a one-dimensional space of rigid rotations.)

We can also consider problems where the control acts on the moment instead of the shear; for example, in (4.2) set $w_{xx}(1,t) = f(t)$ and $w_{xxx}(1,t) = 0$. Again we find that for sufficiently small ρ, the exact controllability is the same as when $\rho = 0$. The proof involves only some minor modifications.

5.2 String equation on flexible boundary. This problem was considered in the compressible case by Micu and Zuazua [8,9]. In the incompressible case the problem can be written

$$w_{tt} + \rho\Lambda_0 w_{tt} - w_{xx} = 0 \qquad \text{on } (0,1)\times\mathbb{R}^+, \tag{5.3}$$

$$w_x(1,t) = 0;\quad w_x(0,t) = f(t) \qquad \text{on } \mathbb{R}^+, \tag{5.4}$$

$$w(x,0) = w^0(x), \qquad w_t(x,0) = w^1(x) \qquad \text{on } (0,1), \tag{5.5}$$

where $\{w^0, w^1\}$ are given in $\mathcal{V}\times\mathcal{H} = H^1(0,1)\times L^2(0,1)$. (We adapt the previous notation to this problem, rather than make new notation.) For convenience, we have set all the physical parameters in (5.3)–(5.5) to 1, other than ρ, the fluid density.

All the steps in the analysis of this problem are the same as in the analysis with the Euler-Bernoulli beam, with the exception that the eigenvalue estimates are weaker, due to the fact that when $\rho = 0$ the spectrum is uniformly spaced. We were able to prove the following result.

Theorem 5.1. *Given any $\{w^0, w^1\} \in \mathcal{V}\times\mathcal{H}$ and $f \in L^2(0,\infty)$ there exists a unique solution w to (5.3)–(5.5) with $w \in C([0,\infty),\mathcal{V})\cap C^1([0,\infty),\mathcal{H})$. Furthermore, there exists $\rho_0 > 0$ such that if $0 \le \rho < \rho_0$ then there exists $T = T(\rho) > 0$ such that (5.3)–(5.5) is exactly controllable on $\mathcal{V}\times\mathcal{H}$ in time T.*

Again, we encounter difficulties if instead of (5.4), fixed end-conditions are imposed.

5.3 Rayleigh beam on flexible boundary.

As a final example, we replace the Euler-Bernoulli beam with a Rayleigh beam:

$$w_{tt} - \gamma w_{xxtt} + \rho\Lambda_0 w_{tt} + w_{xxxx} = 0 \qquad \text{on } (0,1)\times\mathbb{R}^+, \tag{5.6}$$

$$w(0,t) = w(1,t) = w_{xx}(0,t) = 0;$$

$$w_{xx}(1,t) = f(t) \qquad \text{on } \mathbb{R}^+, \tag{5.7}$$

$$w(x,0) = w^0(x), \qquad w_t(x,0) = w^1(x) \qquad \text{on } (0,1), \tag{5.8}$$

where $\{w^0, w^1\}$ are given in $\mathcal{V}\times\mathcal{H} = (H^2(0,1)\cap H^1_0(0,1))\times H^1_0(0,1)$. (Again we adapt the previous notation to this problem.) In (5.6), γ is proportional to the rotational moment of inertia of a beam element and is usually small in comparison to other physical parameters.

In (5.7) we consider the case of *fixed end conditions* with moment control, although our method applies equally well to other boundary conditions.

In the analysis of this problem it is unnecessary to utilize the compactness of Λ_0 due to presence of the term γw_{xxtt}; it is enough to utilize only the continuity of Λ_0. For this reason, boundary conditions involving fixed ends do not present any difficulties. By following the same steps as in the proof of Theorem 1.1 (and slight modifications for the simply supported boundary conditions) we were able to prove the following result.

Theorem 5.2. *Given any $\{w^0, w^1\}\in\mathcal{V}\times\mathcal{H}$ and $f\in L^2(0,\infty)$ there exists a unique solution w to (5.6)–(5.8) with $w\in C([0,\infty),\mathcal{V})\cap C^1([0,\infty),\mathcal{H})$. Furthermore, there exists $\rho_0>0$ such that if $0\le\rho<\rho_0$ then there exists $T=T(\rho)>0$ such that (5.6)–(5.8) is exactly controllable on $\mathcal{V}\times\mathcal{H}$ in time T.*

References

[1] C. Bardos, G. Lebeau, J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary*, SIAM J. Control Optim. 30 (1992), 1024–1065.

[2] H.T. Banks, W. Fang, R.J. Silcox, R.C. Smith, *Approximation methods for the control of acoustic/structure models with piezoceramic actuators*, J. Intelligent Material Systems 4, (1993), 98–116.

[3] H.T. Banks, R.C. Smith, *Well-posedness of a model for structural acoustic coupling in a cavity enclosed by a thin cylindrical shell*, ICASE report 93-10, NASA Langley Research Center (1993).

[4] M. Costabel, *Boundary integral operators on Lipshcitz domains: elementary results*, SIAM J. Math. Anal. 19 (1988), 613–626.

[5] P. Grisvard, *Elliptic Problems in Non Smooth Domains*, Pittman, London, 1985.

[6] J. Lighthill, *An Informal Introduction to Theoretical Fluid Mechanics*, IMA Monograph Series, Clarendon Press, Oxford, 1986.

[7] J.Lighthill, *Energy flow in the cochlea*, J. Fluid Mech. (1981), 149–213.

[8] S. Micu, E. Zuazua, *Propriétés qualitatives d'un modèle hybride bi-dimensionnel intervenant dans le contrôle du bruit*, C.R., Acad. Sci. Paris, 319 (1994), 1263–1268.

[9] S. Micu, E. Zuazua, *Stabilization and periodic solutions of a hybrid system arising in the control of noise*, Proceedings of the IFIP TC7/WG-7.2 International Conference, Laredo, España, Lecture Notes in Pure and Applied Mathematics, 174, Marcel Dekker, New York (1996), 219–230.

[10] J. Nečas, *Les Méthodes directes en théorie des équations elliptiques, Masson, Paris, 1967.*

[11] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag series "Applied Mathematical Sciences " #44 Springer-Verlag, New York 1983.

[12] D.L. Russell, *Nonharmonic Fourier series in the control theory of distributed parameter systems*, J. Math. Anal. Appl. 18 (1967), 542–559.

[13] D.L. Russell, *Controllability and stabilizability theory for partial differential equations. Recent progress and open questions*, SIAM Rev. 20 (1978), 639–739.

[14] J. Sylvester, G. Uhlmann, *The Dirichlet to Neumann map and its applications,* in: *"Inverse Problems in Partial Differential Equations"* (eds. D. Colton, R. Ewing, W. Rundell), SIAM Proceedings Series, Philadelphia, 1990.

[15] G. Weiss, *Admissibility of unbounded control operators*, SIAM J. Control Optim. **27** (1989), 527–545.

[16] R.M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, 1980.

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