Stable manifold Thm Let E be an open subset of $\mathbb{R}^n$, $0 \in E$,

$f \in C^1(E)$, φ_t flow of system $\dot{x} = f(x)$,

$f(0) = 0$ (0 a crit. pt)

$Df(0)$: k eigenval's w/ $Re(\lambda) < 0$

$n-k$ " w/ $Re(\lambda) > 0$

(so $\dot{x} = (Df(0))x$ is "hyperbolic")

Then $\exists$ k dim'l diff'ble manifold S tangent to E^S at 0

$n-k$ " " U " " E^U

for which S is positively invariant w.r.t flow

U " neg. invar. wrt flow

and $\lim_{t \to \infty} \varphi_t(y) = 0 \quad \forall y \in S$

$\lim_{t \to -\infty} \varphi_t(y) = 0 \quad \forall y \in U$

(in sec. 2.8 Perko)

~~2.8~~ ~~2.9~~
~~2.14~~
~~3.1~~ ~~3.2~~

Hartman-Grobman Thm (Same hypothesis as above)

In particular assume 0 is a hyperbolic crit. point of linearized system

$$\dot{x} = Ax \qquad A = Df(0).$$

Then there is a homeomorphism $H : U \to V$, (where U, V are open sets containing 0) and an interval $I_0 \subset \mathbb{R}$ for which

$$H \circ \varphi_t(x_0) = e^{At} H(x_0) \qquad \text{for } t \in I_0$$

i.e trajectories of nonlinear system $\varphi_t(x_0)$ are mapped onto trajectories of linear system, with orientation preserved for $t \in I_0$.

Remark The stable manifold S and unstable manifold U in Stable Manifold Thm are only locally defined ~~near to~~ in a neighborhood of the critical point 0.

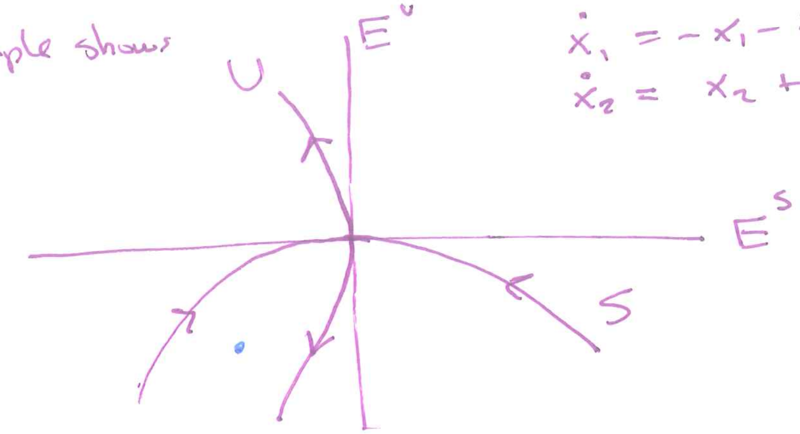
The proof is by using succ. approximations applied to stable and unstable subspaces E^s, E^u of the linearized system.

- the proof provides a way to construct series for S, U

- see ex. 3 p. 111

that example shows

$$\dot{x}_1 = -x_1 - x_2^2$$
$$\dot{x}_2 = x_2 + x_1^2$$



Global stable, unstable manifolds: (at 0) def. by

$$W^s(0) = \bigcup_{t \le 0} \varphi_t(S)$$

$$W^u(0) = \bigcup_{t \ge 0} \varphi_t(U)$$

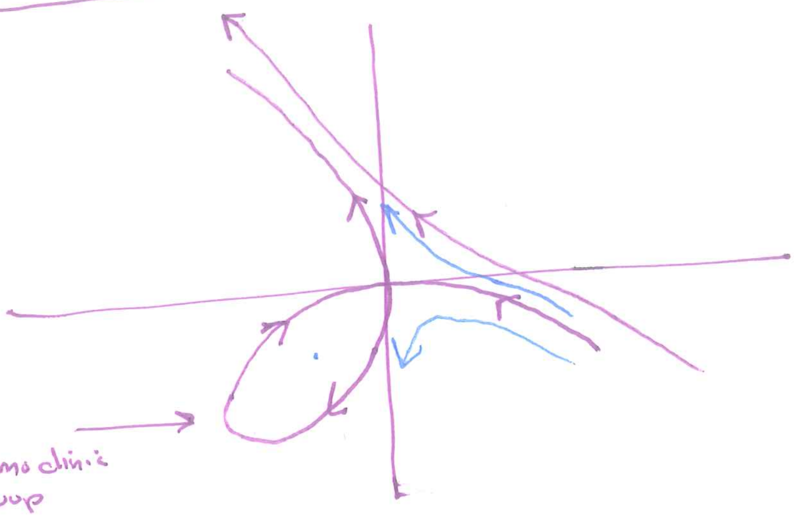
It can be shown: $\forall\, x \in W^s(0)$, $\lim_{t \to \infty} \varphi_t(x) = 0$

$\forall\, x \in W^u(0)$ $\lim_{t \to -\infty} \varphi_t(x) = 0$

Given neighborhood N of 0 in $\mathbb{R}^n$,
the local stable manifold S is

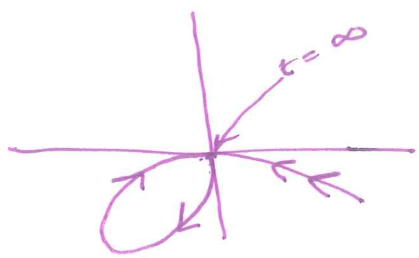
$$S = \{ x \in N : \varphi_t(x) \to 0 \text{ as } t \to \infty \}$$
$$U = \{ x \in N : \varphi_t(x) \to 0 \text{ as } t \to -\infty \}$$

<u>global sol to ex 2</u>

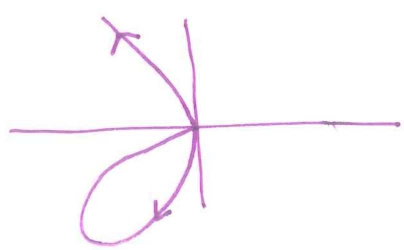


homoclinic loop

$W^s(0)$



$W^u(0)$



<u>Center manifold Theorem</u> (for any crit. point – hyperbolic included)

In addition to $W^s(0)$, $W^u(0)$, corresp. to eigenval's with 0 real part
there is an invariant manifold $W^c(0)$ "center manifold"
tangent to E^c.

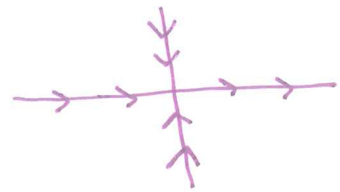
ex 3 $\quad \dot{x}_1 = x_1^2$

$\quad\quad \dot{x}_2 = -x_2$

$$Df(0) = \begin{bmatrix} 2x_1 & 0 \\ 0 & -1 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$\Rightarrow E^s = x_2$ axis

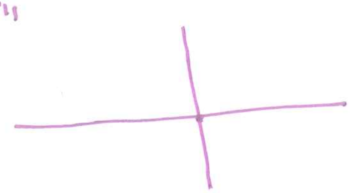
$\quad E^c = x_1$ axis



non lin. $\quad x_1$ axis is invariant

$\quad W^s(0) = x_2$ axis

$\quad W^c(0) =$



$$\frac{dx}{x^2} = dt$$

$$-\frac{1}{x} = t + c$$

$$x = \frac{-1}{t+c}$$

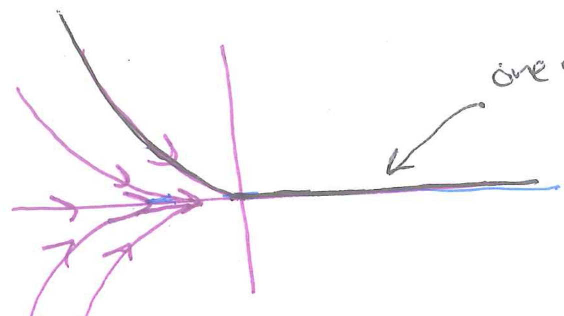
if $x(0) = a$

$$a = \frac{-1}{c} \quad c = -\frac{1}{a}$$

$$x = \frac{-1}{t - 1/a} = \frac{a}{1-at}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{a}{1-at} \\ be^{-t} \end{pmatrix}$$

one center manifold



Any curve $\to 0$ from left of origin, extended by positive x_1 axis is a center manifold.

In particular, center manifolds are not uniquely determined.

See webpage

HW 2.2

~~1 a,b a) skip "Also" part~~

~~b) only do "Also" part~~

~~2.3 1~~

~~2.4 1 a,c~~ 2 b,c (use a DE solver at your own)

~~2.6 1 a,b~~ 2.5 5

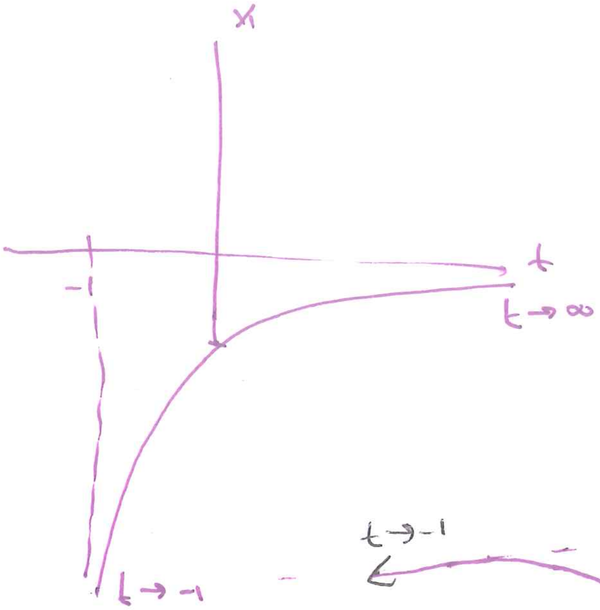
~~2.7 7~~

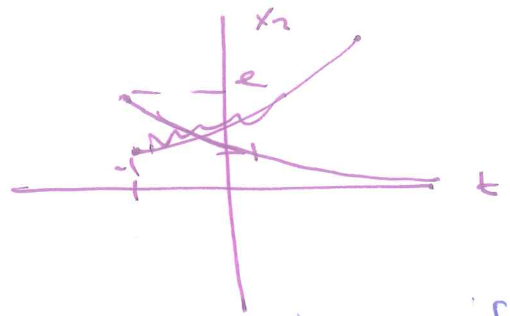
Redo of sol to example 7 p. 116

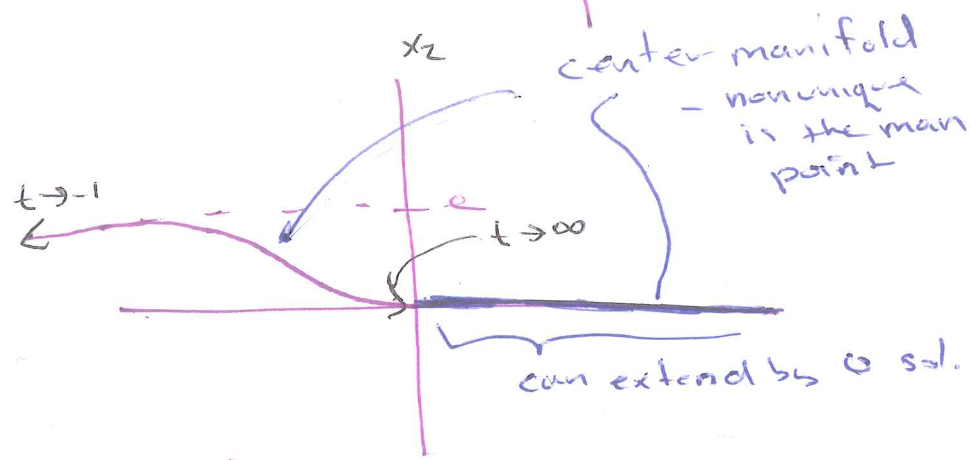
$$\dot{x}_1 = x_1^2 \qquad \dot{x}_2 = -x_2 \Rightarrow x_2 = c_2 e^{-t}$$

$$\Rightarrow x_1 = \frac{c_1}{1 - c_1 t} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{c_1}{1-c_1 t} \\ c_2 e^{-t} \end{pmatrix}$$

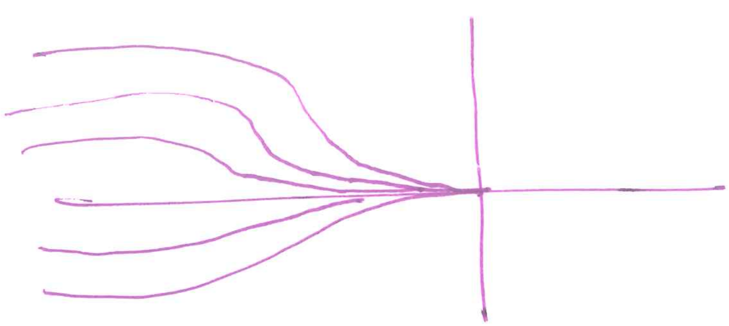
If initial point is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{-1}{1+t} \\ e^{-t} \end{pmatrix}$ then for $t > -1$ sol exists





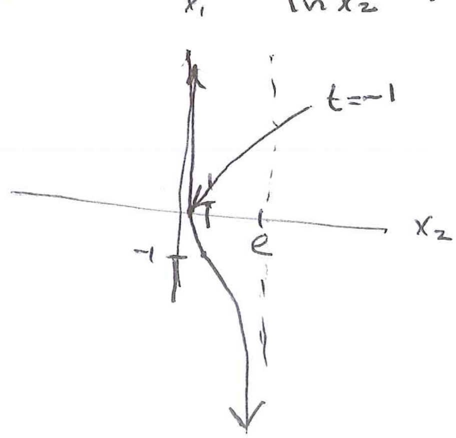


other values besides $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ give similar curves



note: for initial cond $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$x_1 = \frac{-1}{1+t} = \frac{-1}{1 - \ln x_2} = \frac{1}{\ln x_2 - 1}$$



Wan 4.1 Interacting populations

~~simple~~ linear 1 pop. model $\dot{x} = ax$ a growth rate/population

~~simple~~ linear 2 population model

$$\begin{aligned}\dot{x} &= a_{11}x + a_{12}y \\ \dot{y} &= a_{21}x + a_{22}y\end{aligned} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}}_{A}\begin{bmatrix} x \\ y\end{bmatrix}$$

- predator prey

signs of coef's in A: $\begin{bmatrix} + & - \\ + & - \end{bmatrix}$

- competetive $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$

- cooperative $\begin{bmatrix} - & + \\ + & - \end{bmatrix}$

- parasitic $\begin{bmatrix} + & 0 \\ + & - \end{bmatrix}$

5.1 Predator prey

linear
$$\begin{aligned}\dot{x} &= a_{11}x - a_{12}y \\ \dot{y} &= a_{21}x - a_{22}y\end{aligned}$$

nonlinear Lotka-Volterra system: $F(t)$, $R(t)$ denote foxes and rabbits.

$$\left.\begin{aligned}\dot{R} &= aR - \beta R^2 - \overbrace{\gamma}^{a_{12}} RF \\ \dot{F} &= -cF + \delta RF\end{aligned}\right\} = \begin{matrix} f(R,F) \\ g(R,F)\end{matrix}$$

crit. pts

$$\begin{aligned} aR - \beta R^2 - \gamma RF &= 0 \qquad R(a - \beta R - \gamma F) = 0 \\ -cF + \delta RF &= 0 \qquad F(\delta R - c) = 0 \end{aligned}$$

if $F = 0$: $R = 0$ or $a - \beta R = 0 \Rightarrow (0,0)$, $\cancel{(0, \frac{a}{\beta})}$ $(\frac{a}{\beta}, 0)$

if $\delta R - c = 0$ $R = \frac{c}{\delta} \Rightarrow a - \beta\frac{c}{\delta} - \gamma F = 0$ $F = \dfrac{a - \frac{\beta c}{\delta}}{\gamma}$

$$\left(\frac{c}{\delta}, \frac{a - \beta\frac{c}{\delta}}{\gamma}\right)$$

Stability analysis $(\alpha, \beta, c, \delta) = (2, 1, 1, 1)$

$$\dot{R} = 2R - R^2 - RF \qquad \text{crit pts: } (R,F) = (0,0),\ (2,0),\ (1,1)$$

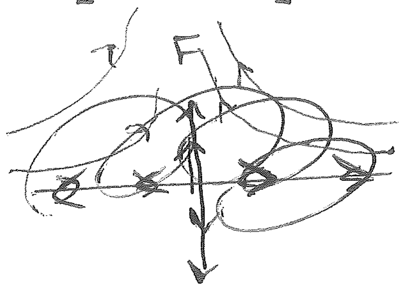
$$\dot{F} = -F + FR$$

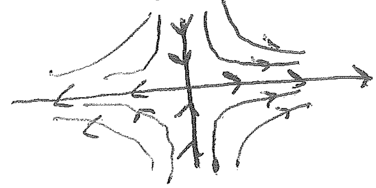
at (0,0)

$$J = \begin{bmatrix} 2-2R-F & -R \\ F & -1+R \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \qquad \lambda = 2 \quad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda = -1 \quad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



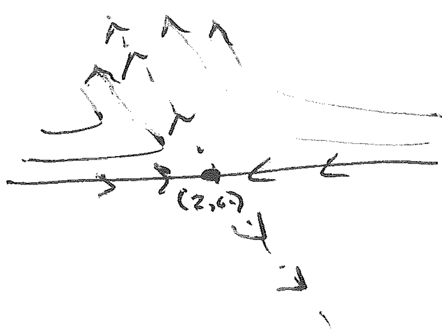


at (2,0)

$$J = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix} \qquad \lambda = -2 \quad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda = 1 \quad \begin{bmatrix} -3 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$



at (1,1) $\quad J = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$

$$(-1-\lambda)(-\lambda) + 1 = 0$$

$$\lambda^2 + \lambda + 1 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

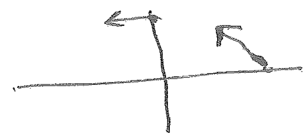
stable spiral pt

Remark: same analysis can be done for "stable" or "unstable" spiral points.

e.g. $A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ $\lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

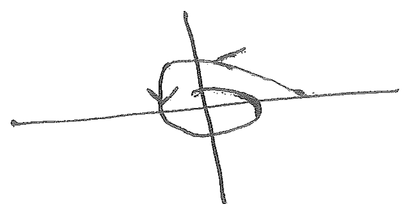
$\dot{x} = -x - y$
$\dot{y} = x$

plug in $(x, y) = (1,0), (0,1)$ to get direction of flow



$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$
$A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$

$\therefore$ spirals counterclockwise



In[1]:=

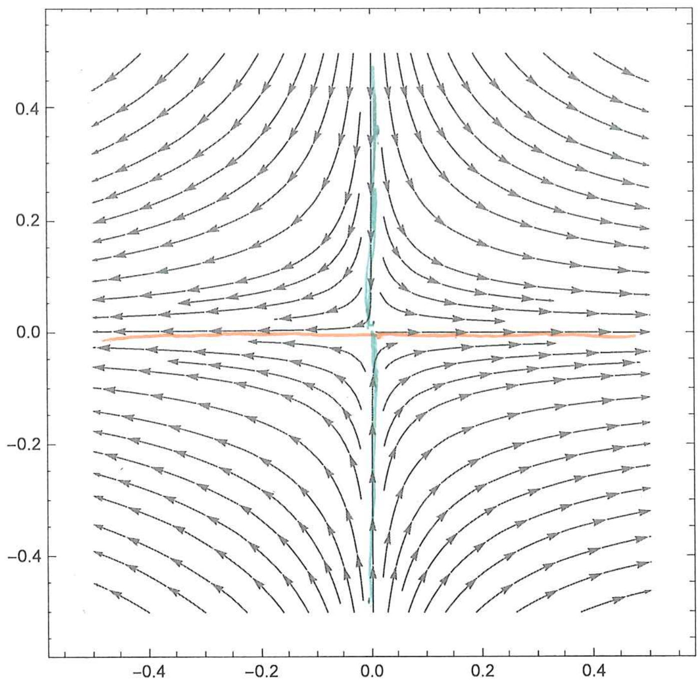
3

Out[1]= 3

In[2]:=

```
StreamPlot[{2 * x - x^2 - x * y, -y + x * y}, {x, -.5, .5}, {y, -.5, .5}]
```

Out[2]=



(0,0)

- stable manifold
- unst. manifold

In[3]:=

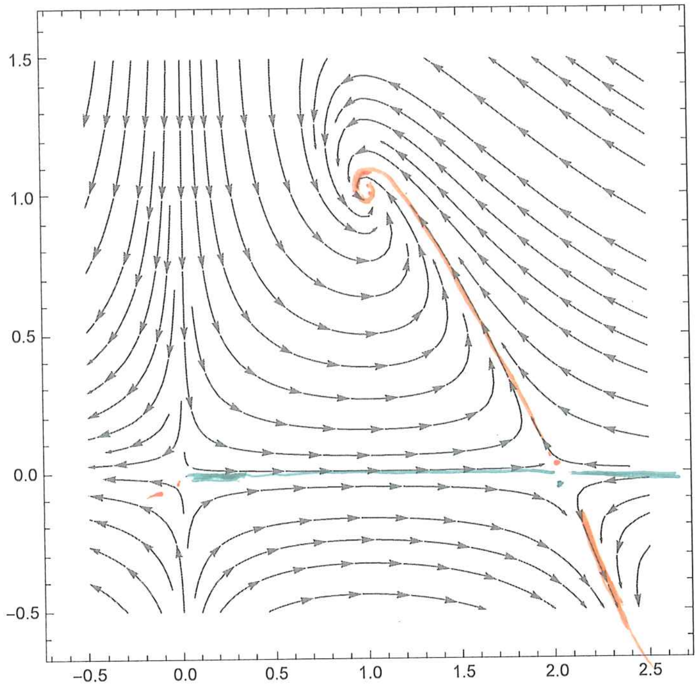
```
Clear[x, y]

StreamPlot[{2 * x - x^2 - x * y, -y + x * y}, {x, 1.5, 2.5}, {y, -.5, .5}]
```

Out[4]= $Aborted

(Dialog) In[19]:=
```
StreamPlot[{2 * x - x^2 - x * y, -y + x * y}, {x, -.5, 2.5}, {y, -.5, 1.5}]
```

(Dialog) Out[19]=



(2,0)

• stable manifold

• unst. man.

(Dialog) In[20]:=

(Dialog) In[21]:=
```
StreamPlot[{2 * x - x^2 - x * y, -y + x * y}, {x, -1, 4}, {y, -2, 2}]
```

(Dialog) Out[21]=