Parameter dependence

Gronwall ineq Suppose $g(t)$ is continuous, real valued, $g(t) \ge 0$ and

$$0 \le g(t) \le C + K\int_0^t g(s)\,ds \qquad t \in [0,a]$$

(K, C positive)

$$\Longrightarrow \quad g(t) \le Ce^{Kt}$$

Thm Let E be an open subset of $\mathbb{R}^n$, $x_0 \in E$, $f \in C^1(E)$

Then $\exists$ $a > 0$, $\delta > 0$ such that

$$\forall\, y \in N_\delta(x_0)$$

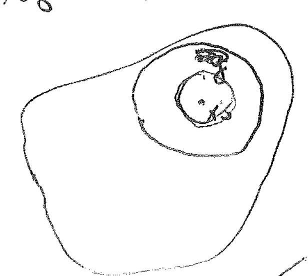
$$\dot{x} = f(x) \qquad x(0) = y \qquad \text{has}$$

a unique sol. $u(t,y)$ with $u \in C^1(G)$

$$G = [-a,a] \times N_\delta(x_0) \subset \mathbb{R}^{n+1}$$

Furthermore $u(t,y)$ is C^2 in t on $[-a,a]$

Repeat same pf, but keep estimates for all $y \in N_\delta(x_0)$:

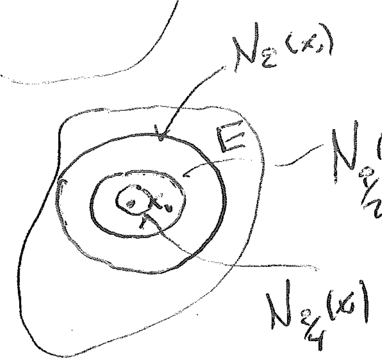


$\varepsilon > 0$: $N_\varepsilon(x_0) \in E$

Let $N_0 = \{x \in \mathbb{R}^n : |x - x_0| \le \varepsilon/2\}$

$\cancel{\delta = \varepsilon/4 \quad \forall\, y \in N_0}$

$\cancel{N_\delta(y) \subset N_0}$



Let $M_0 = \max|f(x)|$ on N_0

$M_1 = \max\|Df(x)\|$ on N_0

Let $\delta = \varepsilon/4$. And for $y \in N_\delta(x_0)$

repeat successive approximations with

$U_0(t,y) = y$

$U_{k+1}(t,y) = y + \int_0^t f(U_k(s,y))\,ds$

However this time in $C(G)$

$$G = [-a,a] \times N_\delta(x_0)$$

note $U_0 \in C(G) \Rightarrow f(U_0(s,y)) \in C(G)$

$$\Rightarrow U_{k+1} \in C(G)$$

as long as

$$\|U_k(t,y) - x_0\|_{C[-a,a]} < \varepsilon/2$$

As last time, we have

$$\|U_{k+1}(t,y) - y\| \leq \int_0^t |f(U_k(s,y))|\,ds \leq M_0 a$$

$$\forall\, t \in [-a,a]$$

and since $y \in N_\delta(x_0)$,

$$\|U_{k+1}(t,y) - x_0\| \leq \|U_{k+1}(t,y) - y + y - x_0\|$$

$$\leq M_0 a + \varepsilon/4 < \frac{\varepsilon}{2} \text{ if}$$

we take $a < \frac{\varepsilon}{4M_0}$

$\Rightarrow$ If $U_k \in C(G) \Rightarrow U_{k+1} \in C(G)$

Also, like proof of Existence + Uniqueness, the iteration is contractive:

$$\|U_{k+1} - U_k\|_{C(G)} \leq (Ka)^k \varepsilon$$

K: Lip constant on N_0 $\Rightarrow$ contractive if $a < 1/K$

$\Rightarrow \quad U(t,y) = y + \int_0^t f(U(s,y))\,ds \qquad (t,y) \in G$

and $U(t,y) \in N_{\varepsilon/2}(x_0) \quad \forall\ (t,y) \in G$

$\Rightarrow \dot U(t,y) = f(U(t,y)) \qquad y(0) = y$

$\Rightarrow \ddot U(t,y) = \underbrace{Df(U(t,y))}_{\in C(G)}\ \underbrace{\dot U(t,y)}_{\in C(G)}$

$\Rightarrow U$ is C^2 in t.

<u>y-dependence</u>

Pick any $y_0 \in N_{\delta/2}(x_0)$, $h \in \mathbb{R}^n$, $|h| < \delta/2$

Let $U(t,y_0)$, $U(t,y_0+h)$ sol's with init data y_0, y_0+h.

$\underbrace{|U(t,y_0+h) - U(t,y_0)|}_{z} = \left| y_0+h + \int_0^t f(U(s,y_0+h))\,ds - y_0 - \int_0^t f(U(s,y_0))\,ds \right|$

$= \left| h + \int_0^t f(U(s,y_0+h)) - f(U(s),y_0)\,ds \right|$

$\le |h| + \int_0^t K\,\underbrace{|U(s,y_0+h) - U(s,y_0)|}_{z}\,ds$

$0 \le z \le |h| + K\int_0^t z\,ds$

Gronwall $\Rightarrow |z| \le |h|\,e^{Kt} \qquad \forall\ t \in [-a,a]$

$|U(t,y_0+h) - U(t,y_0)| \le |h|\,e^{Ka}$

Let $\Phi(t,y_0)$ be fundamental matrix sol. to

$\dot\Phi(t,y_0) = A(t,y_0)\,\Phi \qquad \Phi(0,y_0) = I$

where $A(t,y_0) = Df(U(t,y_0))$

let $g(t) = |U(t,y_0+h) - U(t,y_0) - \Phi(t,y_0)\,h|$

<u>Back:</u> $g(t) \le M_1 \int_0^t g(s)\,ds + \varepsilon_0 |h|\,a\,e^{Ka}$

$\Rightarrow g(t) \le \varepsilon_0 |h|\,a\,e^{Ka} e^{M_1 a} \qquad t \in [-a,a]$, h suff. small

$$\Rightarrow \lim_{|h|\to 0} \frac{|U(t,y_0) - U(t,y_0+h) + \Phi(t,y_0)h|}{|h|} = 0$$

$$\Rightarrow \frac{\partial U}{\partial y}(t,y_0) = \Phi(t,y_0)$$

$\Rightarrow U$ is C^1 on G $\square$

Cor $\quad \Phi(t,y) = \frac{\partial U}{\partial y}(t,y) \quad \forall\ t \in [-a,a]$
$\quad y \in N_\delta(x_0)$

where $\dot{\Phi} = Df[U(t,y)]\,\Phi$

$\Phi(0,y) = I.$

ex $\quad \frac{dU}{dt} = U^2 \quad U(0) = y \quad f(U) = U^2$

$\frac{du}{U^2} = dt \Rightarrow -\frac{1}{U} = t + c \Rightarrow -\frac{1}{y} = c$

$-\frac{1}{U} = t - \frac{1}{y}$

$U = \frac{1}{1/y - t} = \frac{y}{1-yt}$

$$\frac{\partial U}{\partial y} = \frac{(1-yt)\cdot 1 - y(-t)}{(1-yt)^2} = \frac{1}{(1-yt)^2} = \Phi(t,y)$$

$\Phi(0,y) = 1$

$\dot{\Phi} = -2(1-yt)^{-3}\cdot(-y) = \frac{1}{(1-yt)^2}\cdot\frac{2y}{1-yt}$

$$2U\Big|_{U=\frac{y}{1-yt}} = \frac{2y}{1-yt} \Rightarrow \dot{\Phi} = \underbrace{A(t,y)}_{Df[U(t,y)]}\Phi$$

$\Phi(0) = I$

Special case: if $f(x_0) = 0 \Rightarrow u(t, x_0) = x_0 \quad \forall\, t \in \mathbb{R}$

$$\Rightarrow \dot{\Phi} = Df[\underbrace{u(t, x_0)}_{x_0}]\Phi = \underbrace{Df(x_0)}_{A}\Phi \qquad \Phi(0) = I$$

$$\Rightarrow \Phi(t) = e^{At}$$

Thm 2 Let E be an open ~~set~~ subset of $\mathbb{R}^{n+m}$ with $(x_0, \mu_0) \in E$, $f \in C^1(E)$

Then $\dot{x} = f(x, \mu)$

$x(0) = y$

has a unique sol. $u(t, y, \mu)$ with $u \in C^1(G)$

$$G = [-a, a] \times N_\delta(x_0) \times N_\delta(\mu_0)$$

for δ, a suff. small.

Back to Thm I Let E be an open subset of $\mathbb{R}^n$, $x_0 \in E$, $f \in C'(E)$. Then there exists $a > 0$, $\delta > 0$ such that for all $y \in N_\delta(x_0)$

$$\dot{x} = f(x) \qquad x(0) = y$$

has ! sol. $u(t,y)$, with $u \in C'(G)$; $G = [-a,a] \times N_\delta(x_0)$

Furthermore, u is C^2 in t on $[-a,a]$.

Rmk on last part of pf (that shows $u \in C'(G)$ with respect to y)

Let $\Phi(t,y_0)$ be sol. to $\dot{\Phi} = A(t,y_0)\,\Phi$, $\Phi(0,y_0) = I$

where $A = Df(u(t,y_0))$

The proof shows
$$\lim_{|h| \to 0} \frac{|u(t,y_0) - u(t,y_0+h) + \Phi(t,y_0)h|}{|h|} = 0$$

(pf use Gronwall to show $|g(t)| < \varepsilon \quad \forall\, \varepsilon > 0$

where g is $u(t,y_0) - u(t,y_0+h) + \Phi(t,y_0)h$)

This means $\dfrac{\partial u}{\partial y}(t,y_0) = \Phi(t,y_0)$ (and since Φ is C' in y so is u)

where $\dot{\Phi} = Df[u(t,y_0)]\,\Phi \qquad \Phi(0,y) = I$

$(t, y \in G)$

So for example if x_0 is an equilibrium pt. of $\dot{x} = f(x)$

$\Rightarrow f(x_0) = 0 \Rightarrow u(t,x_0) = x_0 \quad \forall\, t \in \mathbb{R}$

$\Rightarrow Df(u(t,x_0)) = Df(x_0)$ (indep. of t)

$\Rightarrow \Phi(t,x_0) = e^{[Df(x_0)]t}$

Example $\quad \dot{x}_1 = -x_1 \quad\Rightarrow\quad x_1 = c_1 e^{-t}$

$\dot{x}_2 = -x_2 + x_1^2 \quad\Rightarrow\quad \dot{x}_2 + x_2 = (c_1 e^{-t})^2 = c_1^2 e^{-2t}$

$$\underbrace{e^{t}(\dot{x}_2 + x_2)}_{(x_2 e^t)^{\cdot}} = c_1^2 e^{-t}$$

$$\Rightarrow\quad x_2 e^t = c_2 + \frac{c_1^2 e^{-t}}{-1} \Rightarrow x_2 = c_2 e^{-t} - c_1^2 e^{-2t}$$

if $x(0) = y \qquad c_1 = y_1$

$x_2(0) = y_2 = c_2 - c_1^2 = c_2 - y_1^2$

$\Rightarrow\quad c_2 = y_2 + y_1^2$

$$\Rightarrow\quad U(t,y) = \begin{pmatrix} y_1 e^{-t} \\ (y_2 + y_1^2)e^{-t} - y_1^2 e^{-2t} \end{pmatrix}$$

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an equilibrium pt

$$Df(0) = \begin{bmatrix} -1 & 0 \\ 2x_1 & -1 \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow\quad \Phi(t,0) = e^{[Df(0)]t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

also $$\frac{\partial U}{\partial y}(t,0) = \begin{bmatrix} e^{-t} & 0 \\ 2y_1 e^{-t} - 2y_1 e^{-2t} & e^{-t} \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} e^{-t} & \\ & e^{-t} \end{bmatrix}$$ ✓

(Thus, using $\Phi(t,0)$
we don't need to know
sol's to the DE
to compute $\frac{\partial U}{\partial y}(t,0)$)

Now consider $\dot{x} = f(x, \mu)$ $x(0) = y$ $\in N_\delta(x_0) \subset \mathbb{R}^n$

parameter in $N_\delta(\mu_0) \subset \mathbb{R}^m$

let $G = [-a, a] \times N_\delta(x_0) \times N_\delta(\mu_0)$ $\subset \mathbb{R}^{n+m}$

Assuming $f \in C^1(E)$, $(x_0, \mu_0) \in E$, $\exists\, a>0, \delta>0$

so sol is $C^1(G)$.

Maximal interval of existence and uniqueness

$\dot{x} = f(x)$ $x(0) = x_0$ $f \in C^1(E)$

E-U Thm $\Rightarrow$ E-U holds at least on $(-a, a)$

Thm 1 There exists a maximal E-U interval $J = (\alpha, \beta)$. It is open.

(possibly $\alpha = -\infty$, $\beta = \infty$).

Thm 2 Assume E-U interval is $J = (\alpha, \beta)$.

If $\beta < \infty$, then given any compact subset K of E, there exists $t \in (\alpha, \beta)$ such that

$$x(t) \notin K$$

Thm 3 If $\beta < \infty$, $[0, \beta)$ is the "right maximal E-U interval" there exists $t \in [0, \beta)$ such that

$$x(t) \notin K \quad (\text{for any } K \text{ compact in } E)$$

Cor If $\lim_{t \to \beta^-} x(t) = x_1$ (if limit exists)

then $x_1 \in$ boundary (E)

Cor 2 Let $[0, \beta)$ be right maximal E-U interval.

If there exists K compact $\subset E$ such that

$$\{x(t) : t \in [0, \beta)\} \subset K$$

then $\beta = \infty$

Examples

1) $\dot{x} = x^2 \quad x(0) = 1 \quad \Longrightarrow \quad x(t) = \frac{1}{(1-t)^2}$

$(\alpha, \beta) = (-\infty, 1)$

$E = \mathbb{R}$

as $t \to 1^-$ $\quad x(t) \to \infty$

2) $\dot{x} = -\frac{1}{2x} \quad x(0) = 1 \quad \Longrightarrow \quad x(t) = \sqrt{1-t}$

$(\alpha, \beta) = (-\infty, 1)$

$E = (0, \infty)$

as $t \to 1^-$ $\quad x(t) \to 0 \in \dot{E}$

Flow of a DE.

1. linear case.

Consider $\dot{x} = Ax$. Let $\varphi_t = e^{At}$. Then

(i) $\varphi_0(x) = x$ all $x \in \mathbb{R}^n$

(ii) $\varphi_s(\varphi_t(x)) = \varphi_{s+t}(x)$ $s, t \in \mathbb{R}$, $x \in \mathbb{R}^n$

(iii) $\varphi_{-t}(\varphi_t(x)) = \varphi_t(\varphi_{-t}(x)) = x$ $t \in \mathbb{R}$

Def Let E be open in $\mathbb{R}^n$, $f \in C^1(E)$. For $x_0 \in E$,
let $\varphi(t, x_0)$ be sol. to $\dot{x} = f(x)$ $x(0) = x_0$
on $I(x_0)$ = maximal interval of existence

each t defines a mapping $\varphi(t, x_0) : E \to E$.

The family $\{\varphi(t, x_0)\}_{t \in I(x_0)}$ is the flow of the DE

Rmk: By some rescaling of time and space, can assume that for all $x_0 \in E$, $I(x_0) = \mathbb{R}$
$\Rightarrow \varphi_t \in C^1(E)$
and $\varphi(t, x) \in C^1(\mathbb{R} \times E)$
so (i), (ii), (iii) hold

can show

- $\varphi_{s+t}(x_0) = \varphi_s(\varphi_t(x_0))$ if $x_0 \in E$, $t \in I(x_0)$, $s \in I(\varphi_t(x_0))$

- $\varphi_{-t}(\varphi_t(x)) = x$ for all $x \in U$

  $\varphi_t(\varphi_{-t}(y)) = y$ " " $y \in V$

Def 2 (p. 99 Perko) Let $E \subset \mathbb{R}^n$, $f \in C^1(E)$, $\varphi_t : E \to E$ flow of DE $\dot{x} = f(x)$

i) $S \subset E$ is invariant wrt flow φ_t if $\varphi_t(S) \subset S$ all $t \in \mathbb{R}$

ii) " positively invariant " ... if $\varphi_t(S) \subset S$ all $t \geq 0$

iii " neg. invariant if $\varphi_t(S) \subset S$ all $t \leq 0$

Linearization Consider $\dot{x} = f(x)$

Behavior near critical points if $f(x_0) = 0$,

let $A = Df(x_0)$.

Then Ax is the linear part of $f(x)$ at x_0

Def $x_0 \in \mathbb{R}^n$ is an equil. pt or crit. pt if $f(x_0) = 0$.

Crit. pt. x is hyperbolic $\mathrm{Re}\,\lambda_i \neq 0$ $\forall$ eigenvalues λ_i of A.

$\dot{x} = Ax$ is the linearization of $\dot{x} = f(x)$ at x_0

If $x_0 = 0$ (so $f(0) = 0$) by Taylor thm

$$f(x) = \underbrace{Df(0)x}_{Ax} + \frac{1}{2} D^2 f(0)(x, x) + \dots$$

Note: If $\varphi_t : E \to \mathbb{R}$ is flow of $\dot{x} = f(x)$, $f(x_0) = 0$

$\Rightarrow \{x_0\}$ is an invariant set. (called a zero)

or a singular point of the vector field

if x_0 is a crit. pt:

- "saddle" if all λ have $\mathrm{Re}\,\lambda < 0$
- "source" " " " " $\mathrm{Re}\,\lambda > 0$
- "saddle" if at least 1 λ_i with $\mathrm{Re}\,\lambda_i > 0$

  " " " λ_j " $\mathrm{Re}\,\lambda_j < 0$

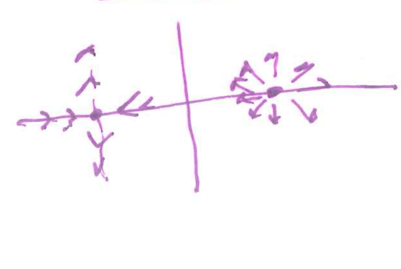
and x_0 hyperbolic

ex. $f(x) = \begin{bmatrix} x_1^2 - x_2^2 - 1 \\ 2x_2 \end{bmatrix}$ $\Rightarrow$ crit pts $(1,0)$, $(-1,0)$

~~$Df\begin{pmatrix}1\\0\end{pmatrix}$~~ $Df = \begin{bmatrix} 2x_1 & -2x_2 \\ 0 & 2 \end{bmatrix}$ $Df(1,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

source

$Df\begin{pmatrix}-1\\0\end{pmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ saddle



ex Let E^u and E^s be the unstable and stable manifolds of $\dot{x} = Ax$

(e.g. E^s = span of gen. eigenvectors corresp to λ_i with $Re(\lambda_i) > 0$)

then if $x_0 \in E^s$, $t \in \mathbb{R}$ $e^{At} x_0 \in E^s$

$\therefore$ E^s is invariant under flow $\varphi_t = e^{At}$

E^u " " " also

E^c is also

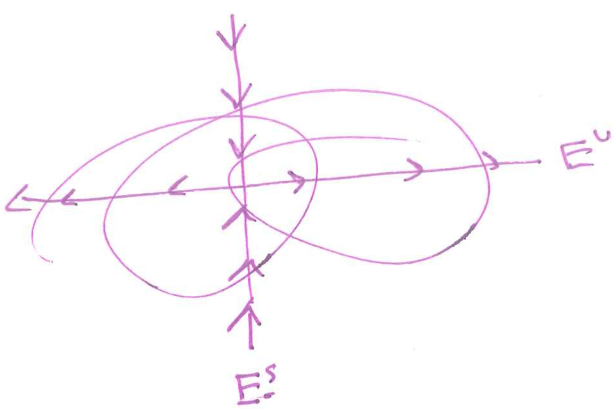
ex2 $f(x) = \begin{bmatrix} -x_1 \\ x_2 + x_1^2 \end{bmatrix}$

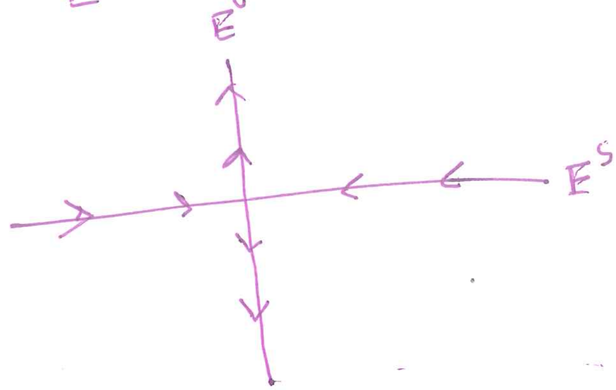
$$\varphi_t(y) = \begin{bmatrix} y_1 e^{-t} \\ y_2 e^{t} + \frac{y_1^2}{3}\left(e^{t} - e^{-2t}\right) \end{bmatrix}$$

linear analysis crit. pts $(0,0)$

linearization $\begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$ so $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x$

saddle pt. eigen pairs: $(-1, \underbrace{\text{span}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\substack{\text{stable s.s.} \\ E^s}})$ $(1, \underbrace{\text{span}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{E^u})$





Non lin. system

note that if $x_2 = -x_1^2$ $y \in \text{span}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ i.e $y = \begin{bmatrix} y_1 \\ 0 \end{bmatrix}$

then $\varphi_t(y) = \begin{bmatrix} y_1 e^{-t} \\ y_2 \\ \frac{y_1^2}{3}(e^{t} - e^{-2t}) \end{bmatrix}$

$y \in \text{span}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ i.e $y = \begin{bmatrix} 0 \\ y_2 \end{bmatrix} \Rightarrow \varphi_t(y) = \begin{bmatrix} 0 \\ y_2 e^{t} \end{bmatrix} \in \text{span}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore$ E^U is also invariant for non lin. system

setting $\varphi_t(y) = \alpha \varphi_0(y) = \alpha y$ gives

$\alpha \begin{bmatrix} -y_1 \\ y_2 + y_1^2 \end{bmatrix} = \begin{bmatrix} y_1 e^{-t} \\ y_2 e^{t} + \frac{y_1^2}{3}(e^{t} - e^{-2t}) \end{bmatrix}$

top eq: $\alpha = -e^{-t}$

$-e^{-t}(y_2 + y_1^2) = y_2 e^{t} + \frac{y_1^2}{3}(e^{t} - e^{-2t})$

$y_2(-e^{-t} - e^{t}) = y_1^2\left(e^{-t} + \frac{e^{t}}{3} - \frac{e^{-2t}}{3}\right)$?

if $y_1 = 0$

$\Rightarrow \varphi_t\begin{pmatrix} 0 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ y_2 e^{t} \end{pmatrix}$

$\in \text{span}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ span $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an invariant set.

Let $S = \{y \in \mathbb{R}^2 : y_2 = -y_1^2/3\}$

$\Rightarrow \varphi_t(y) = \begin{bmatrix} y_1 e^{-t} \\ -\frac{y_1^2}{3} e^{-2t} \end{bmatrix} = \begin{bmatrix} y_1 e^{-t} \\ -\frac{(y_1 e^{-t})^2}{3} \end{bmatrix} \in S$

$\therefore$ S is invariant