Ch. 2 linear systems $\dot{x} = Ax$

has a unique sol. through $x_0 \in \mathbb{R}^n$:

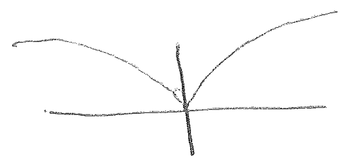
$$x(t) = e^{At}x_0$$

Non lin. theory $\dot{x} = f(x)$ $f: E \subset \mathbb{R}^n \to \mathbb{R}^n$

(E open set in $\mathbb{R}^n$)

A couple examples to keep in mind:

1) $\dot{x} = 3x^{2/3}$ $x(0) = 0$



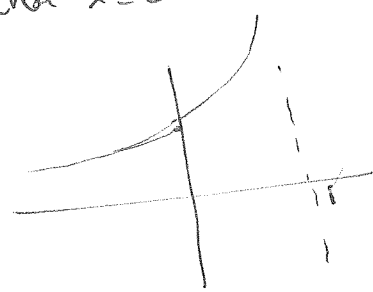
has sol's $u(t) = t^3$

and $v(t) \equiv 0$

$\therefore$ $f'(0)$ DNE and nonuniqueness could occur when $x = 0$

2) $\dot{x} = x^2$ $x(0) = 1$

$\Rightarrow$ $x(t) = \dfrac{1}{1-t}$



sol. exists, is unique on $(-\infty, 1)$

Def $f: \mathbb{R}^n \to \mathbb{R}^n$ is diff'ble at $x_0 \in \mathbb{R}^n$ if $\exists$ $n \times n$ matrix $Df(x_0)$ ($\in L(\mathbb{R}^n)$) such that

$$\lim_{|h| \to 0} \frac{|f(x_0+h) - f(x_0) - (Df(x_0))h|}{|h|} = 0$$

$Df(x_0)$ is the (full) derivative.

And $Df(x_0)$ applied to x is $(Df(x_0))x = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x_0)x_j$

$$Df = \left[\frac{\partial f_i}{\partial x_j}\right]$$: $n \times n$ Jacobian

ex $\quad f(x) = \begin{bmatrix} x_1 \sin x_2 \\ x_1 x_2 \end{bmatrix}$

$$Df = \begin{bmatrix} \sin x_2 & x_1 \cos x_2 \\ x_2 & x_1 \end{bmatrix}$$

$$Df(3,\pi) = \begin{bmatrix} 0 & -3 \\ \pi & 3 \end{bmatrix}$$ note $(0,0)$ is only crit. pt.

ex if $f(x) = Ax$ $\quad A$ $n \times n$ matrix

$$Df = A$$

Continuity If $F : V_1 \to V_2$ $\quad V_1, V_2$ Normed Vect Spaces

F contin. at $x_0 \in V_1$ iff $\forall \varepsilon > 0 \ \exists \delta > 0$:

$\|x - x_0\|_1 < \delta \ (x \in V_1) \Rightarrow \|F(x) - F(x_0)\|_2 < \varepsilon$

Def $f : E \to \mathbb{R}^n \in C^1(E)$ iff f is diff'ble on E

and $Df : E \to L(\mathbb{R}^n)$ is contin. on E

Thm. $f \in C^1(E)$ iff each term $\dfrac{\partial f_i}{\partial x_j} \in C^1(E)$

Similarly $f \in C^k(E)$ [i.e. $D^k f$ exists, is contin. on E]

iff all kth order partials

$$\frac{\partial^k f}{\partial x_{j_1} x_{j_2} \cdots x_{j_k}} \quad \text{exist, are contin.}$$

f is <u>analytic</u> if each component of f (f_j) has a convergent taylor series on E.

Formula for D^2: $\quad D^2 f(x_0) : E \times E \to \mathbb{R}^n$

$$D^2 f(x_0)(x,y) = \sum_{j,k=1}^{n} \frac{\partial^2 f(x_0)}{\partial x_j \, \partial x_k} x_j y_k$$

<u>ex</u> $\quad f(x) = \begin{bmatrix} x_1 \sin x_2 \\ x_1 x_2 \end{bmatrix}$

$$D^2 f(0,0)(x,y) = \begin{pmatrix} 0 x_1 y_1 + (\cos 0) x_1 y_2 + (\cos 0) x_2 y_1 + 0 x_2 y_2 \\ 0 x_1 y_1 + 1 x_1 y_2 + 1 x_2 y_1 + 0 x_2 y_2 \end{pmatrix}$$

$$= \begin{bmatrix} x_1 y_2 + x_2 y_1 \\ x_1 y_2 + x_2 y_1 \end{bmatrix}$$

<u>Existence + Uniqueness</u> $\quad$ If $f \in C(E)$ $\quad$ (E open subset of $\mathbb{R}^n$)

$x(t)$ is a sol. on interval I if

$\cancel{\dot{x}(t) \in f(x(t))}$ on I $\quad$ (and all terms involved are well defined)

$\dot{x}(t) = f(x(t))$

<u>Def</u> $\quad f : E \to \mathbb{R}^n$ is Lip on E $\exists$ $K > 0$:

$$|f(x) - f(y)| \le K |x - y| \quad \forall\, x, y \in E$$

f is <u>locally Lip</u> if $\forall\, x_0 \in E$ there is an ε-nbhd : $N_\varepsilon(x_0)$ $\quad$ ($= x \in \mathbb{R}^n : |x - x_0| < \varepsilon$)

for which f is lip. on $N_\varepsilon(x_0)$

ex $f(x) = x^2$ Not Lip on $\mathbb{R}$

since $|f(x+1) - f(x)| = (x+1)^2 - x^2 = 2x+1$

not $\le \underbrace{k\,|x-y|}_{=K}$

on $E = \mathbb{R}$

but y^2 is locally Lip.

(take $N_\varepsilon(x_0) = N_1(x_0)$)

Lemma Let E be open subset of $\mathbb{R}^n$. Then if

$f \in C'(E)$ f is locally Lip on E

pf in $\mathbb{R}^1$ Let $x_0 \in E$, pick any $\delta > 0$ so

$N_{2\delta}(x_0) \subset E \implies \overline{N}_\delta(x_0) \subset E$

f' cont. on $\overline{N}_\delta(x_0) \implies f'$ bdd on $N_\delta(x_0)$

If $x_1, x_2 \in N_\delta(x_0)$

$|f(x_2) - f(x_1)| = |f'(t)|\,|x_2 - x_1|$

t between x_1, x_2

by MVT

$\implies$ locally Lip.

The space $C(I)$. (continuous functions on interval I)

define $\|u\| = \sup_I |u(t)|$

ex $\|x^n\|_{C(0,1)} = 1 \quad \forall n$

$\|x^n\|_{(0,1/2)} = (1/2)^n \quad \forall n$

Fact $C(I)$ is complete; i.e. Every Cauchy seq. converges

– see def. 3 –

ex Solve $\dot{x} = ax \quad x(0) = x_0$ by successive approx's

$\int_0^t \dot{x} = \int_0^t ax \Rightarrow x(t) - x(0) = \int_0^t a x(t)\,dt$

or $x(t) = x_0 + \int_0^t a x(s)\,ds$.

For iteration, use
$$x_{n+1}(t) = x_0 + \int_0^t a x_n(s)\,ds$$

start with $u_0(t) = x_0$

$$u_1(t) = x_0 + \int_0^t a u_0(s)\,ds = x_0 + a\left[x_0 s\Big|_0^t\right]$$
$$= x_0 + a x_0 t = x_0(1+at)$$

$$u_2(t) = x_0 + \int_0^t a x_0(1+as)\,ds$$
$$= x_0 + a x_0 t + \frac{a^2 t^2}{2} x_0 = x_0\left(1 + at + \frac{a^2t^2}{2!}\right)$$

$\vdots$

$$u_k(t) = x_0 \underbrace{\left(1 + at + \frac{a^2t^2}{2!} + \dots + \frac{a^k t^k}{k!}\right)}_{p(t)}$$

can be shown that $\|p_k(t) - e^{at}\|_{C(I)} \to 0$ as $k \to \infty$

for any finite interval I

Def A fixed point of mapping $K : C \subseteq X \to C$ is an element $x \in C$ such that

$$K(x) = x.$$

Moreover, K is a contraction if $\exists\ \theta \in [0,1)$ s.t.:

$$\|K(x) - K(y)\| \le \theta \|x - y\| \quad \forall\ x, y \in C$$

Notation: $K^0(x) = x$, $K^n(x) = K(K^{n-1}(x))$

Thm 2.1 Teschl Let C be a nonempty closed subset of Banach sp. X and let $K : C \to C$ be a contraction, then K has a unique fixed point $\bar{x} \in C$ such that

$$\|K^n(x) - \bar{x}\| \le \frac{\theta^n}{1-\theta} \|K(x) - x\| \quad x \in C$$

in particular, $K^n(x) \to \bar{x}$ in X

Rmk: Consider DE $\dot{x} = f(x)$ $x(0) = x_0$

$$\int_0^t \dot{x}(s)\,ds = \int_0^t f(x(s))\,ds$$

$$\Rightarrow x(t) - x_0 = \int_0^t f(x(s))\,ds$$

$$\text{or } x(t) = \underbrace{x_0 + \int_0^t f(x(s))\,ds}_{K(x)}$$

so X could be continuous functions on an an interval $[-a, a]$.

C could be functions in X which are bounded by some M: i.e., $|f(t)| \le M$ $\forall\ t \in [-a, a]$

TP can show fixed point thm applies to K

$\Longrightarrow$ sol's exist and are unique.

A seq. $\{U_n\}$ in normed vector space V is <u>contractive</u> if $\|U_{n+1} - U_n\| < \rho \|U_n - U_{n-1}\|$ $n \geq 2$.

<u>Prop.</u> $\{U_n\}$ contractive $\Longrightarrow$ $\{U_n\}$ Cauchy

<u>sketch</u> $\|U_{m+p} - U_m\|$

$$= \|U_{m+p} - U_{m+p-1} + U_{m+p-1} + U_{m+p-2} \cdots + U_{m+1} - U_m\|$$

$$\leq \sum_{k=1}^{p} \|U_{m+k} - U_{m+k-1}\|$$

$$= \|U_{m+p} - U_{m+p-1}\| + \cdots + \|U_{m+1} - U_m\|$$

$$\leq (\rho^{p-1} + \cdots + \rho + 1)\|U_{m+1} - U_m\|$$

$$\leq \frac{1}{1-\rho}\|U_{m+1} - U_m\| \leq \underbrace{\frac{1}{1-\rho} \cdot \rho^{m-1}\|U_2 - U_1\|}_{< \varepsilon}$$

$\therefore$ Given ε, pick m :

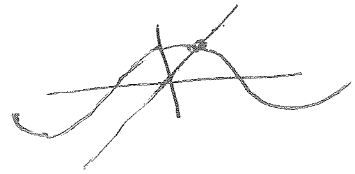
then for any $n, n+p \geq m$

$\|U_{n+p} - U_n\| < \varepsilon$

Also $C[I]$ is complete $\therefore$ contractive seq's in $C[I]$ are convergent

<u>Thm</u> Let E be an open subset of $\mathbb{R}^n$ that contains x_0 and assume $f \in C^1(E)$. Then $\exists$ $a > 0$: $\dot{x} = f(x)$ $x(0) = x_0$ has a ! sol. on $[-a, a]$

Fixed point examples



ex to solve $x = \cos x$,

let $K(x) = \cos x$.

Then $|K(x) - K(y)| = |\cos x - \cos y| = |(-\sin c)(x-y)|$

c betw x, y

if we use $|\sin c| \le 1 \Rightarrow |K(x) - K(y)| \le |x-y|$
(not contractive).

However, if $C = [-.9, .9]$, then

if $\alpha \in C$ $|\cos\alpha| \le \cos .9 \Rightarrow \cos x \in C$

also $|\sin\alpha| < |\alpha| \le .9$

$\Rightarrow K : C \to C$ and is contraction.

$\Rightarrow$ $x^* = \lim_{n\to\infty} x_n$ $\quad$ $x_0 = .5$, $x_{n+1} = \cos x_n$

i.e. $x_n = \cos(\cos(\cos(\cdots(\cos .5)) \to x^*$

ex $f(x) + \frac{1}{2}\int_0^x t f(t)\,dt = 2$

$$\Rightarrow f(x) = \underbrace{2 - \tfrac{1}{2}\int_0^x t f(t)\,dt}_{K(f)}$$

$$|K(v) - K(u)| = \left|\tfrac{1}{2}\int_0^x t v(t) - t u(t)\,dt\right|$$

$$\le \tfrac{1}{2}\int_0^{|x|} |t(v(t) - u(t))|$$

Try to solve in $C[0,1]$

$$\le \tfrac{1}{2}\int_0^{|x|} t\, \|v-u\|_{C[0,1]}\,dt$$

$$\le \tfrac{1}{2}\cdot\frac{|x|^2}{2}\|v-u\|$$

$$\le \tfrac{1}{4}\|v-u\|$$

Existence-Uniqueness Thm Let E be an open subset of $\mathbb{R}^n$, $x_0 \in E$, $f \in C^1(E)$. Then $\exists\ a > 0$:

$$\dot{x} = f(x) \qquad x(0) = x_0$$

has a ! solution $x(t)$ on $[-a, a]$.

Pf $f \in C^1(E) \Rightarrow$ locally Lip in E.

$\Rightarrow \exists\ \varepsilon > 0$ with $N_\varepsilon(x_0) \subset E$ and

$$|f(x) - f(y)| \le K|x-y| \qquad x, y \in N_\varepsilon(x_0)$$

Pick $b < \varepsilon$ Then if

$$N_0 = \{x \in \mathbb{R}^n : |x - x_0| \le b\}, \quad f \text{ is bdd on } N_0$$

Let $M = \max_{x \in N_0} |f(x)|$

let $U_0(t) = x_0$

$$U_{k+1}(t) = x_0 + \int_0^t f(U_k(s))\,ds \qquad k = 0, 1, \ldots$$

want to find interval $[-a, a]$ where $\{U_k(t)\}$ is contractive

estimate
$$|U_{k+1}(t) - x_0| \le \left|\int_0^t f(U_k(s))\,ds\right|$$
$$\le \int_0^t |f(U_k(s))|\,ds$$

if $U_k \in N_0\ \forall\, t \in [-a, a]$

then (above) $\le \int_0^t M\,ds = Mt \le Ma$

$\Rightarrow U_{k+1} \in N_0$ also if $Ma < b$

$\therefore$ pick $a < \frac{b}{M}$ then

$$U_k \in N_0\ \forall\, t \in [-a, a] \Rightarrow U_{k+1} \in N_0\ \forall\, t \in [-a, a]$$

math induction $\Rightarrow U_k(t)$ is contin, $U_k(t) \in N_0\ \forall\, t \in [-a, a]$ for all $k \in \mathbb{N}$

Then $|U_2(t) - U_1(t)| \le \int_0^t |f(U_1(s)) - f(U_0(s))|\, ds$

$$\le \int_0^t K\,|U_1(s) - U_0(s)|\, ds$$

$$\le K \int_0^t \|U_1 - U_0\|\, ds$$

$$\le Ka\, \|U_1 - U_0\|$$

$$\Rightarrow \|U_2 - U_1\| \le Ka\|U_1 - U_0\|$$

$$|U_{j+1}(t) - U_j(t)| \le \int_0^t |f(U_j(s)) - f(U_{j-1}(s))|\, ds$$

$$\le K \int_0^t |U_j(s) - U_{j-1}(s)|\, ds$$

$$\le Ka\|U_j - U_{j-1}\|$$

$\therefore$ contractive if a is reduced below $1/K$

$\Longrightarrow \{U_k\}$ convergent in $C[-a, a]$

$U(t) = \lim_{k \to \infty} U_k(t)$ conv. unif on $[a, a]$

(U is contin. since $C[-a, a]$ is complete)

Furthermore

$$\underbrace{U_{k+1}(t)}_{\downarrow\, U(t)} = \underbrace{x_0}_{\downarrow\, x_0} + \underbrace{\int_0^t f(U_k(s))\, ds}_{\downarrow\, \int_0^t f(U(s))\, ds}$$

$\therefore\ U(t) = x_0 + \int_0^t f(U(s))\,ds$

i.e. $\dot{U} = f(U(t))$ $\quad U(0) = x_0$

unif. convergence $\Rightarrow U \in N_0$ also

so sol. holds on $[-a, a]$

Uniqueness if U, V both are sol'ns

$$U(t) = x_0 + \int_0^t f(U(s))\,ds$$
$$V(t) = x_0 + \int_0^t f(V(s))\,ds$$

$$\|U-V\| \le \max_{[-a,a]} |U(t) - V(t)|$$ (assume occurs at $t_1 \in [-a,a]$)

$$= \left|\int_0^{t_1} f(U(s)) - f(V(s))\,ds\right|$$
$$\le \left|\int_0^{t_1} |f(U(s)) - f(V(s))|\,ds\right|$$
$$\le \left|\int_0^{t_1} K\,|U(s) - V(s)|\,ds\right|$$
$$\le \left|\int_0^{t_1} K\,\|U-V\|\,ds\right|$$
$$\le aK\,\|U-V\|$$

$$\Rightarrow \|U-V\|(1-aK) \le 0$$
$$\Rightarrow \|U-V\| \le 0 \Rightarrow U = V$$

or ? $U = K(U)$ and $V = K(V)$, K contraction

$$\|U-V\| = \|K(U) - K(V)\|$$
$$\le \rho\,\|U-V\|$$
$$\Rightarrow \|U-V\|(1-\rho) = 0 \qquad \rho < 1 \Rightarrow U = V$$