Exact boundary controllability of a Rao-Nakra sandwich beam

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ABSTRACT

We consider a three layer Rao-Nakra sandwich beam with damping proportional to shear included in the core layer. We prove that eigenvectors of the beam form a Riesz basis for the natural energy space. In the damped case, we are able to give precise conditions under which solutions decay at a uniform exponential rate. We also consider the problem of boundary control using bending moment and lateral force control at one end. We prove that the space of exact controllability has finite co-dimension and provide sufficient conditions (related to small damping) for exact controllability to a zero energy state.

Keywords: Sandwich beam; exact controllability, boundary control

1. INTRODUCTION

The classical Rao-Nakra[1] sandwich beam model consists of two outer "face plate" layers (which are assumed to be relatively stiff) which "sandwich" a much more compliant "core layer". The Rao-Nakra model is derived using Euler-Bernoulli beam assumptions for the face plate layers, Timoshenko beam assumptions for the core layer and a "no-slip" assumption for the displacements along the interface. The following are the equations of motion[2] :

$$\begin{array}{rcll} m\ddot{w} - \alpha D_x^2 \ddot{w} + K D_x^4 w - D_x N h_2 (G_2 \varphi + \tilde{G}_2 \dot{\varphi}) & = & 0 & \text{on } (0,L)\times(0,\infty) \\ \mathbf{h}_{\mathcal{O}} \mathbf{p}_{\mathcal{O}} \ddot{\mathbf{v}}_{\mathcal{O}} - \mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} D_x^2 \mathbf{v}_{\mathcal{O}} + \mathbf{B^T} (G_2 \varphi + \tilde{G}_2 \dot{\varphi}) & = & 0 & \text{on } (0,L)\times(0,\infty) \end{array} \tag{1}$$

where $\varphi = h_2^{-1} \mathbf{B} v_{\mathcal{O}} + N w_x$. In addition we consider the following controlled boundary conditions:

$$\begin{array}{rcll} w(0,t) = D_x^2 w(0,t) = D_x \mathbf{v}_{\mathcal{O}}(0,t) = w(L,t) & = & 0 & t>0, \\ D_x^2 w(L,t) = M(t),\ D_x \mathbf{v}_{\mathcal{O}}(L,t) = \mathbf{g}_{\mathcal{O}}(t) & & & t>0 \end{array}. \tag{2}$$

In the above, w denotes the transverse displacement of the beam, φ denotes the shear angle of the core layer, $\mathbf{v}_{\mathcal{O}} = (v_1, v_3)^T$ is the vector of longitudinal displacement along the neutral axis of the outer layers. ($i = 1, 3$ is for the outer layers, $i = 2$ is for the core layer.) The density of the ith layer is denoted ρ_i, the thickness h_i, the Young's modulus E_i, the shear modulus of the core layer is G_2. We let $m = \sum h_i \rho_i$ denote the mass density per length, $\alpha = \rho_1 h_1^3/12 + \rho_3 h_3^3/12$ is a moment of inertia parameter, $K = E_1 h_1^3/12 + E_3 h_3^3/12$ is the bending stiffness. In addition,

$$\mathbf{p}_{\mathcal{O}} = \text{ diag}\,(\rho_1, \rho_3), \quad \mathbf{h}_{\mathcal{O}} = \text{ diag}\,(h_1, h_3), \quad \mathbf{E}_{\mathcal{O}} = \text{ diag}\,(E_1, E_3)$$

$$\mathbf{B} = (-1, 1), \qquad N = \frac{h_1 + h_2 + h_3}{h_2}.$$

The boundary control functions acting at the right end of the beam are $M(t)$, the applied moment, and $\mathbf{g}_{\mathcal{O}}(t) = (g_1(t), g_3(t))^T$, the longitudinal force. Our main result is the following:

Theorem 1. *The eigenvectors associated with (1), (2) form a Riesz basis for the finite energy space* $X_0 \times X_1$.

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The finite energy space is defined in Section 2.

We also prove several consequences of Theorem 1. In particular, the semigroup associated with (1) decays at a rate that is determined by the spectrum. Consequently, by analyzing the eigenvalues we are able to give precise conditions under which energy of solutions decays exponentially to zero. (See Proposition 2, Corollary 1).

Finally, we consider the problem of controlling an initial finite energy state to another in time T with controls $\mathbf{g}_{\mathcal{O}}$, $M(t)$ belonging to $L^2(0,T)$. We prove that if $T > \tau$ where

$$\tau = 2L\left[\min\left(\sqrt{\frac{K}{\alpha}}, \sqrt{\frac{E_1}{\rho_1}}, \sqrt{\frac{E_3}{\rho_3}}\right)\right]^{-1}, \tag{3}$$

then the system (1)–(2) is exactly controllable modulo a finite dimensional quotient. If the damping $\tilde{G}_2$ is sufficiently small, this finite dimensional quotient consists of the space determined by "zero energy" uncontrollable state $w = 0, (v_1, v_3) = (1, 1)$ (See Theorem 4 and Corollary 2).

Remark 1. This paper can be considered as a continuation of Hansen and Rajaram,[3] where specifically the case of distinct wave speeds $\sqrt{\frac{E_1}{\rho_1}}, \sqrt{\frac{E_3}{\rho_3}}$ is considered. Here we focus on the case in which these two wave speeds are the same.

The paper is organized as follows. In Section 2 we describe the semigroup formulation of (1), (2) with $M(t) = 0$, $\mathbf{g}_{\mathcal{O}} = 0$. In Section 3 we prove Theorem 1. In Section 4 we analyze the moment problem and in Section 5 we prove controllability results.

2. SEMIGROUP FORMULATION

Let $(u, v)_\Omega = \int_0^L u \cdot \bar{v}\, dx$, where u may be either scalar or vector valued. Define quadratic forms a and c by

$$\begin{aligned} c(w, \mathbf{v}_{\mathcal{O}}) &= (mw, w)_\Omega + \alpha(w_x, w_x)_\Omega + (\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\mathbf{v}_{\mathcal{O}}, \mathbf{v}_{\mathcal{O}})_\Omega \\ a(w, \mathbf{v}_{\mathcal{O}}) &= K(w_{xx}, w_{xx})_\Omega + (\mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}\mathbf{v}_{\mathcal{O}x}, \mathbf{v}_{\mathcal{O}x})_\Omega + (G_2 h_2 \varphi, \varphi)_\Omega. \end{aligned} \tag{4}$$

The energy of the beam is given by

$$\mathcal{E}(t) = \frac{R}{2}(c(\dot{w}, \dot{\mathbf{v}}_{\mathcal{O}}) + a(w, \mathbf{v}_{\mathcal{O}}))$$

where R is the width of the beam. Let $U = (u, \mathbf{u})^T := (w, \mathbf{v}_{\mathcal{O}})^T$, $V = (v, \mathbf{v})^T := (\dot{w}, \dot{\mathbf{v}}_{\mathcal{O}})^T$, $Y = (U, V)$. Also define $J : H^2(0, L) \cap H_0^1(0, L) \to L^2(0, L)$ by $J\theta = m\theta - \alpha D_x^2\theta$. The first order form of (1) with M and $\mathbf{g}_{\mathcal{O}}$ set to zero is

$$\frac{dY}{dt} = \mathcal{A}Y := \begin{pmatrix} 0 & I \\ A_1 & A_2 \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix}, \tag{5}$$

$$\text{where } A_1 U = \begin{pmatrix} J^{-1}(-KD_x^4 u + D_x N h_2 G_2[h_2^{-1}(\mathbf{B}\mathbf{u} + h_2 N D_x u)] \\ \mathbf{h}_{\mathcal{O}}^{-1}\mathbf{p}_{\mathcal{O}}^{-1}[\mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}D_x^2\mathbf{u} - \mathbf{B}^T G_2[h_2^{-1}(\mathbf{B}\mathbf{u} + h_2 N D_x u)]] \end{pmatrix} \tag{6}$$

$$A_2 V = \begin{pmatrix} J^{-1}(D_x N h_2 \tilde{G}_2[h_2^{-1}(\mathbf{B}\mathbf{v} + h_2 N D_x v)] \\ \mathbf{h}_{\mathcal{O}}^{-1}\mathbf{p}_{\mathcal{O}}^{-1}[-\mathbf{B}^T\tilde{G}_2[h_2^{-1}(\mathbf{B}\mathbf{v} + h_2 N D_x v)]] \end{pmatrix}. \tag{7}$$

The energy inner product is defined by

$$< Y, \hat{Y} >_e = a(U; \hat{U}) + c(V; \hat{V}), \qquad (\hat{Y} = (\hat{U}, \hat{V})),$$

where $a(\cdot\,;\cdot)$ and $c(\cdot\,;\cdot)$ are the bilinear forms that coincide with the previously defined quadratic forms $a(\cdot)$, $c(\cdot)$ on the diagonal. Let

$$\begin{aligned} X_1 &= \{u, \mathbf{u}\} \in H^2(0, L) \cap H_0^1(0, L) \times (H^1(0, L))^2 \\ X_0 &= \{u, \mathbf{u}\} \in H_0^1(0, L) \times (L^2(0, L))^2. \end{aligned}$$

It can be shown[2] that the equations of motion are well-posed on the energy space $(U, \dot{U}) \in C([0,T]; X_1 \times X_0)$. It is not hard to prove the same for semigroup solutions. The domain of this semigroup is $\mathcal{D}(\mathcal{A}) = X_2 \times X_1$, where

$$X_2 = \{(u, \mathbf{u}) \in X_1 : u \in H^3(0,L),\ \mathbf{u} \in (H^2(0,L))^2 + BC's\}$$

where "+BC's" means $D_x^2 u$ and $D_x \mathbf{u}$ vanish at each end.

Theorem 2. *Let $\mathcal{A}$ and $\mathcal{D}(\mathcal{A})$ be as above. Then $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to X_1 \times X_0$ is the generator of a C_0 dissipative semigroup on $X_1 \times X_0$.*

One may formulate the weak equations of motion as follows:

$$\frac{d}{dt}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 & I \\ A_1 & A_2 \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{B}\{M, \mathbf{g}_{\mathcal{O}}\} \end{pmatrix}; \tag{8}$$

where

$$\mathcal{B}\{M, \mathbf{g}_{\mathcal{O}}\} = \begin{pmatrix} J^{-1}M(t)\delta_L'(x) \\ \mathbf{h}_{\mathcal{O}}^{-1}\mathbf{p}_{\mathcal{O}}^{-1}\mathbf{g}_{\mathcal{O}}\delta_L(x) \end{pmatrix}. \tag{9}$$

In order to define solutions of (8) one first extends the semigroup $e^{\mathcal{A}t}$ to a weaker space defined by duality. However for the inputs defined in (8)-(9), we will later see that (8) is well posed on $X_1 \times X_0$.

3. SPECTRAL ANALYSIS OF $\mathcal{A}$

Setting $\mathcal{A}Y = \lambda Y$ is equivalent to

$$V = \lambda U, \qquad A_1 U + A_2 \lambda U = \lambda^2 U. \tag{10}$$

3.1. The case $\lambda = 0$

If $\lambda = 0$ then $V = 0$ and we need to find all solutions to $A_1 U = 0$, or equivalently,

$$KD_x^4 u - D_x N h_2 G_2 \varphi = 0 \tag{11}$$

$$-\mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}} D_x^2 \mathbf{u} + \mathbf{B}^T G_2 \varphi = 0 \tag{12}$$

$$\mathbf{B}\mathbf{u} + h_2 N D_x u = h_2 \varphi. \tag{13}$$

We multiply (12) from the left by $\mathbf{B}\mathbf{h}_{\mathcal{O}}^{-1}\mathbf{E}_{\mathcal{O}}^{-1}$ and use (13) to obtain

$$-D_x^2\varphi + PG_2\varphi = -h_2 N D_x^3 u; \qquad P = \mathbf{B}(\mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}})^{-1}\mathbf{B}^T > 0. \tag{14}$$

It follows that $(-D_x^2 + PG_2)$ is a positive, invertible operator on $\mathcal{D}(\mathcal{A})$ and (11) becomes

$$KD_x^4 u - DNh_2 G_2(-D_x^2 + PG_2)^{-1}(-h_2 N D_x^3 u) = 0. \tag{15}$$

It follows from positivity of the differential operator in (14) (and the boundary conditions for u) that $u = 0$. Hence (14) implies $\varphi = 0$. Thus (13) implies $\mathbf{B}\mathbf{u} = 0$. It follows that $\mathbf{u} = (a + bx)\vec{1}_{\mathcal{O}}$ ($\vec{1}_{\mathcal{O}} = (1,1)^T$) is the general form of $\mathbf{u}$. However, considering the boundary conditions, we conclude the null space is the following

$$\mathbf{u} = \vec{1}_{\mathcal{O}}, \qquad u = 0, \qquad V = 0. \tag{16}$$

Associated with this null vector, one also has a generalized eigenvector of the form

$$U = 0, \qquad v = 0, \qquad \mathbf{v} = \vec{1}_{\mathcal{O}}. \tag{17}$$

3.2. Eigenvectors with $u = 0$

When $u = 0$ we are led to the following system

$$\begin{aligned} D_x N(G_2 \mathbf{B}\mathbf{u} + \tilde{G}_2 \mathbf{B}\lambda \mathbf{u}) &= 0 && (18)\\ \lambda^2 \mathbf{h}_O \mathbf{p}_O \mathbf{u} - \mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} D_x^2 \mathbf{u} + \mathbf{B}^T h_2^{-1}(G_2 \mathbf{B}\mathbf{u} + \tilde{G}_2 \mathbf{B}\lambda \mathbf{u}) &= 0. && (19)\end{aligned}$$

Let $\mathbf{z} = D_x \mathbf{u}$. Applying D_x to (19) leads to

$$\lambda^2 \mathbf{h}_O \mathbf{p}_O \mathbf{z} - \mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} D_x^2 \mathbf{z} = 0. \tag{20}$$

Since $\mathbf{z}$ satisfies $\mathbf{z}(0) = \mathbf{z}(1) = 0$, in order to have nontrivial solutions, we must have that the wave speeds $\sqrt{\frac{E_1}{\rho_1}}$ and $\sqrt{\frac{E_3}{\rho_3}}$ are the same. In this case, let $\mu = \sqrt{\frac{E_1}{\rho_1}} = \sqrt{\frac{E_3}{\rho_3}}$. Equation (20) becomes

$$(\lambda^2 - \mu^2 D_x^2)\mathbf{z} = \mathbf{0}$$

which leads to the following eigenvalues and eigenvectors :

$$\lambda = \pm\, i\mu \frac{k\pi}{L}, u = 0,\ \mathbf{u} = (1,1)^T \cos(\frac{k\pi x}{L}), k = 1, 2, 3, ... \tag{21}$$

In the case of distinct wave speeds, (20) implies $\mathbf{z} = 0$, in which case $\mathbf{u}$ is constant in each component. Then (18) is satisfied and (19) reduces to

$$\lambda^2 \mathbf{h}_O \mathbf{p}_O \mathbf{u} + \mathbf{B}^T h_2^{-1}(G_2 \mathbf{B}\mathbf{u} + \tilde{G}_2 \mathbf{B}\lambda \mathbf{u}) = 0. \tag{22}$$

This system is equivalent to the following matrix system:

$$\begin{pmatrix} 0 & I \\ \mathbf{h}_{\mathcal{O}}^{-1}\mathbf{p}_{\mathcal{O}}^{-1}\mathbf{B}^T \frac{G_2}{h_2}\mathbf{B} & \mathbf{h}_{\mathcal{O}}^{-1}\mathbf{p}_{\mathcal{O}}^{-1}\mathbf{B}^T \frac{\tilde{G}_2}{h_2}\mathbf{B} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}. \tag{23}$$

The above system, being of order 4, has 4 eigenvalues (up to multiplicity). Two of these correspond to $\lambda = 0$, with $\mathbf{u}$ and $\mathbf{v}$ given by the null vector and generalized null vector as described in (16) and (17). The other two roots satisfy

$$\lambda^2 + \lambda R\tilde{G}_2/h_2 + RG_2/h_2 = 0, \quad R = \mathbf{B}\mathbf{h}_{\mathcal{O}}^{-1}\mathbf{p}_{\mathcal{O}}^{-1}\mathbf{B}^T > 0, \tag{24}$$

where each λ is associated with an eigenvector of the form

$$U = (0, h_3\rho_3, -h_1\rho_1)^T, V = \lambda U. \tag{25}$$

In the case that $\tilde{G}_2^2 R/h_2 = 4G_2$, $\lambda = -\tilde{G}_2 R/(2h_2)$ is a double root and a corresponding generalized eigenvector can be found.

3.3. All other eigenvectors

The second equation in (10) can be written as

$$\begin{aligned} m\lambda^2 u - \alpha\lambda^2 D_x^2 u + K D_x^4 u - D_x N h_2 (G_2 + \lambda\tilde{G}_2)\varphi &= 0 && (26)\\ \mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\lambda^2 \mathbf{u} - \mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}} D_x^2 \mathbf{u} + \mathbf{B}^T (G_2 + \lambda\tilde{G}_2)\varphi &= 0 && (27)\\ \mathbf{B}\mathbf{u} + h_2 N D_x u &= \varphi. && (28)\end{aligned}$$

We look for solutions of the form

$$u = \frac{a}{\sigma_k}\sin\sigma_k x, \qquad \mathbf{u} = \vec{C}_k \cos\sigma_k x, ; \qquad \sigma_k = k\pi/L, \quad \vec{C}_k = (c_1, c_3)^T. \tag{29}$$

Solutions of this form satisfy all the homogeneous boundary conditions. We seek λ_k, a and $\vec{C}_k$ so that the system (26)–(28) is satisfied. For simplicity we omit the subscript k. Upon substitution of the modal solutions (29) into (26)–(28) we obtain

$$\frac{a}{\sigma}m\lambda^2 + \frac{a}{\sigma}\alpha\lambda^2\sigma^2 + \frac{a}{\sigma}K\sigma^4 + \sigma N h_2(G_2 + \lambda\tilde{G}_2)(\frac{\mathbf{B}}{h_2}\vec{C} + \frac{a}{\sigma}N\sigma) = 0 \quad (30)$$

$$\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\lambda^2\vec{C} + \mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}\sigma^2\vec{C} + \mathbf{B}^T(G_2 + \lambda\tilde{G}_2)(\frac{\mathbf{B}}{h_2}\vec{C} + \frac{a}{\sigma}N\sigma) = 0. \quad (31)$$

The system (30)-(31) can be expressed as a standard eigenvalue problem for a six-by-six matrix, and hence there exists for each $\sigma > 0$ six linearly independent eigenvectors and generalized eigenvectors. Define $\Gamma(\lambda) = G_2/\lambda + \tilde{G}_2$. In the case of distinct wave speeds $\sqrt{\frac{E_1}{\rho_1}} \neq \sqrt{\frac{E_3}{\rho_3}}$, it has been shown[3] that $\forall k \in \mathbb{N}$ there are six eigenvalues occurring in complex conjugate pairs $\{\lambda^{\pm}_{k,j}\}; j = 0, 1, 3$, where

$$\lambda^{\pm}_{k,0} = -\frac{N\tilde{G}_2 h_2}{2\alpha} \pm i\sigma_k\sqrt{\frac{K}{\alpha}} + \mathcal{O}(k^{-1}) \quad (32)$$

$$\lambda^{\pm}_{k,j} = -\frac{\tilde{G}_2}{2h_2 h_j \rho_j} \pm i\sigma_k\sqrt{\frac{E_j}{\rho_j}} + \mathcal{O}(k^{-1}),\ j = 1, 3. \quad (33)$$

Associated eigenvectors (and possibly generalized eigenvectors) Y_λ are of the form

$$Y_\lambda = \begin{pmatrix} \frac{U_\lambda}{\lambda} \\ U_\lambda \end{pmatrix};\ U_\lambda = \begin{pmatrix} u_{k,j} \\ \mathbf{u}_{k,j} \end{pmatrix} = \begin{pmatrix} \frac{A_{k,j}}{\sigma_k}\sin(\sigma_k x) \\ \vec{B}_{k,j}\cos(\sigma_k x) \end{pmatrix}, \quad (34)$$

where k sufficiently large the $Y_{\lambda^{\pm}_{k,j}}$ are eigenvectors and

$$\begin{pmatrix} A_{k,0} \\ \vec{B}_{k,0} \end{pmatrix} = \begin{pmatrix} 1 \\ \mathcal{O}(k^{-1}) \\ \mathcal{O}(k^{-1}) \end{pmatrix}, \begin{pmatrix} A_{k,1} \\ \vec{B}_{k,1} \end{pmatrix} = \begin{pmatrix} \mathcal{O}(k^{-1}) \\ 1 \\ \mathcal{O}(k^{-1}) \end{pmatrix}, \begin{pmatrix} A_{k,3} \\ \vec{B}_{k,3} \end{pmatrix} = \begin{pmatrix} \mathcal{O}(k^{-1}) \\ \mathcal{O}(k^{-1}) \\ 1 \end{pmatrix}. \quad (35)$$

3.4. Analysis for equal wave speeds

Now let us consider the case where the wave speeds are the same. Again we look for eigenvectors of the form (29). In the case where $\sqrt{\frac{E_1}{\rho_1}} = \sqrt{\frac{E_3}{\rho_3}}$, we find that $\sqrt{\frac{K}{\alpha}}$ also coincides with those wave speeds. Letting $\mu = \sqrt{\frac{K}{\alpha}} = \sqrt{\frac{E_1}{\rho_1}} = \sqrt{\frac{E_3}{\rho_3}}$ we find that (30),(31) becomes

$$(m\lambda^2 + \alpha\lambda^2\sigma^2 + \alpha\mu^2\sigma^4 + \sigma N h_2(G_2 + \lambda\tilde{G}_2)N\sigma)\frac{a}{\sigma} + \sigma N(G_2 + \lambda\tilde{G}_2)\mathbf{B}\vec{C} = 0 \quad (36)$$

$$\mathbf{B^T}(G_2 + \lambda\tilde{G}_2)N\sigma\frac{a}{\sigma} + (\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}(\lambda^2 + \mu^2\sigma^2) + \frac{(G_2 + \lambda\tilde{G}_2)}{h_2}\mathbf{B}\mathbf{B^T})\vec{C} = 0. \quad (37)$$

Letting $y = \frac{\lambda}{\sigma}$ and dividing (36) and (37) by σ^4 and σ^2, respectively, we obtain

$$(m\frac{y^2}{\sigma^2} + \alpha(y^2 + \mu^2) + \frac{1}{\sigma^2}N^2 h_2(\frac{G_2}{\sigma} + y\tilde{G}_2))a + \frac{1}{\sigma^2}N(\frac{G_2}{\sigma} + y\tilde{G}_2))\mathbf{B}\vec{C} = 0$$

$$\frac{1}{\sigma}\mathbf{B^T}(\frac{G_2}{\sigma} + y\tilde{G}_2))Na + (\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}(y^2 + \mu^2) + \frac{(\frac{G_2}{\sigma} + y\tilde{G}_2)))}{h_2\sigma^2}\mathbf{B^T}\mathbf{B})\vec{C} = 0.$$

Letting $m' = \dfrac{m}{G_2}, \vec{v} = (a, c_1, c_3)^T$, and $\Lambda = \text{ diag}(\alpha, h_1\rho_1, h_3\rho_3)$, the above system becomes

$$(y^2+\mu^2)\Lambda\vec{v} + \frac{y\tilde{G}_2}{\sigma}\begin{pmatrix} N^2h_2 & N\mathbf{B} \\ N\mathbf{B^T} & \frac{1}{h_2}\mathbf{B^T}\mathbf{B}\end{pmatrix}\vec{v} + \frac{G_2}{\sigma^2}\begin{pmatrix} m'y^2+N^2h_2 & N\mathbf{B} \\ N\mathbf{B^T} & \frac{1}{h_2}\mathbf{B^T}\mathbf{B}\end{pmatrix}\vec{v} = 0. \tag{38}$$

We need to choose y in (38) to balance the powers of σ^{-1} as $\sigma \to \infty$. Consequently, $y = \pm i\mu + \frac{r}{\sigma}$. Since we know that roots occur in complex conjugate pairs, we consider the case

$$y = i\mu + \frac{r}{\sigma}. \tag{39}$$

Then (38) becomes the following

$$\frac{i\mu}{\sigma}\left[\begin{pmatrix} 2r\alpha & 0 & 0 \\ 0 & 2rh_1\rho_1 & 0 \\ 0 & 0 & 2rh_3\rho_3\end{pmatrix} + \tilde{G}_2\begin{pmatrix} N^2h_2 & N\mathbf{B} \\ N\mathbf{B^T} & \dfrac{1}{h_2}\mathbf{B^T}\mathbf{B}\end{pmatrix}\right]\vec{v} = \mathcal{O}(\sigma^{-2}).$$

It follows that the bracketed term above is singular and we obtain the eigenvalue problem

$$(2r\Lambda + \tilde{G}_2\begin{pmatrix} N^2h_2 & N\mathbf{B} \\ N\mathbf{B^T} & \dfrac{1}{h_2}\mathbf{B^T}\mathbf{B}\end{pmatrix})\vec{v} = 0. \tag{40}$$

This eigenvalue problem can be converted to a standard eigenvalue problem by multiplying the left and right sides with $\Lambda^{-\frac{1}{2}}$. It is then easy to compute the corresponding values of $r, \vec{v}$. We find that the pairs $\{r, \vec{v}\}$ are given by the following:

$$\{r_0, \vec{v}_0\} = \left\{0, \begin{pmatrix} 0 \\ 1 \\ 1\end{pmatrix}\right\}, \{r_1, \vec{v}_1\} = \left\{0, \begin{pmatrix} 1 \\ Nh_2/2 \\ -Nh_2/2\end{pmatrix}\right\} \tag{41}$$

$$\{r_3, \vec{v}_3\} = \left\{\frac{-\tilde{G}_2}{2}(\frac{N^2h_2}{\alpha} + \frac{1}{h_2}(\frac{1}{h_1\rho_1} + \frac{1}{h_3\rho_3})), \begin{pmatrix} -Nh_2 \\ \alpha/h_2h_1\rho_1 \\ -\alpha/h_2h_3\rho_3\end{pmatrix}\right\}. \tag{42}$$

One can use Rouche's theorem to verify the validity of (39) for each of the calculated eigenpairs in (41). Let $y_j,\ j = 0, 1, 3$ denote y in (39) with values of r_j in (41), (42). The actual form of the eigenvalues and eigenvectors of $\mathcal{A}$ are given by

$$\lambda^+_{k,j} = y_j\sigma_k = r_j + i\mu\sigma_k + \mathcal{O}(k^{-1}),\ \lambda^-_{k,j} = \overline{\lambda^+_{k,j}} \tag{43}$$

$$Y = (U, V), U = \frac{1}{\lambda}\begin{pmatrix} u \\ \mathbf{u}\end{pmatrix}, V = \lambda U, \tag{44}$$

where $u, \mathbf{u}$ corresponding to $\lambda^+_{k,j}$ are given by

$$\begin{pmatrix} u_{k,j} \\ \mathbf{u}_{k,j}\end{pmatrix} = \begin{pmatrix} (1/\sigma_k)\sin\sigma_k x & & \\ & \cos\sigma_k x & \\ & & \cos\sigma_k x\end{pmatrix}(\vec{v}_k + \mathcal{O}(k^{-1})) \tag{45}$$

where $\vec{v}_k$'s are as in (41),(42).

Proposition 1. *The spectrum of $\mathcal{A}$ consists of the eigenvalues*

$$\sigma(\mathcal{A}) = \bigcup_{k=0}^{\infty} S_k,$$

where S_0 consists of the double eigenvalue 0 and the roots of (24) and for $k \in \mathbb{N}$,

$$S_k = \{\lambda^+_{k,0}, \lambda^-_{k,0}, \lambda^+_{k,1}, \lambda^-_{k,1}, \lambda^+_{k,3}, \lambda^-_{k,3}\},$$

where $\lambda^+_{k,j}, \lambda^-_{k,j}$ are complex conjugate roots which are given by (32), (33) in the case of distinct wave speeds, and (43) in the case of identical wave speeds. For $k = 0$, eigenvectors and generalized eigenvectors are given in (16)-(17) and (25). The eigenvectors and generalized eigenvectors corresponding to $\lambda = \lambda^\pm_{k,j}$ (for k sufficiently large) are given by

$$Y_\lambda = \begin{pmatrix} \frac{U_\lambda}{\lambda} \\ U_\lambda \end{pmatrix}; \; U_\lambda = \begin{pmatrix} u_{k,j} \\ \mathbf{u}_{k,j} \end{pmatrix}; \begin{pmatrix} u_{k,j} \\ \mathbf{u}_{k,j} \end{pmatrix} = \begin{pmatrix} A_{k,j}\sin(\sigma_k x) \\ \vec{B}_{k,j}\cos(\sigma_k x) \end{pmatrix}, \tag{46}$$

where in the case of identical wave speeds, the eigenvectors are described by (44) and (45), and in the case of distinct wave speeds,

$$\begin{pmatrix} A_{k,0} \\ \vec{B}_{k,0} \end{pmatrix} = \begin{pmatrix} 1 \\ \mathcal{O}(k^{-1}) \\ \mathcal{O}(k^{-1}) \end{pmatrix}, \begin{pmatrix} A_{k,1} \\ \vec{B}_{k,1} \end{pmatrix} = \begin{pmatrix} \mathcal{O}(k^{-1}) \\ 1 \\ \mathcal{O}(k^{-1}) \end{pmatrix}, \begin{pmatrix} A_{k,3} \\ \vec{B}_{k,3} \end{pmatrix} = \begin{pmatrix} \mathcal{O}(k^{-1}) \\ \mathcal{O}(k^{-1}) \\ 1 \end{pmatrix}. \tag{47}$$

In either case, the eigenvectors are block orthogonal with respect to blocks of eigenvectors corresponding to the eigenvalues in S_k, $k \in \mathbb{N}$.

Proof of Theorem 1: Denote by $\tilde{\mathcal{A}}$ the generator $\mathcal{A}$ in the undamped case, i.e., with $\tilde{G}_2 = 0$. For $k \in \mathbb{N}$ let $\left(\tilde{\lambda}^\pm_{k,0}, \tilde{\lambda}^\pm_{k,1}, \tilde{\lambda}^\pm_{k,3}\right)$ and $\left(\tilde{Y}^\pm_{k,0}, \tilde{Y}^\pm_{k,1}, \tilde{Y}^\pm_{k,3}\right)$ be the associated eigenvalues and eigenvectors. For $k = 0$ the spectrum of $\tilde{\mathcal{A}}$ consists of the double eigenvalue 0 and the roots of (18)-(19) with parameter $\tilde{G}_2$ set to 0. The corresponding eigenvectors and generalized eigenvectors are again described by (16)-(17) and (25). It is easy to see that the span of these eigenvectors is the same in the damped and undamped cases. Denote by E_0 the span of these eigenvectors. It is enough to show that the span of the eigenvectors of $\mathcal{A}$ corresponding to $k \in \mathbb{N}$ form a Riesz basis for the orthogonal complement of E_0. First note that $\tilde{\mathcal{A}}$ is skew hermitian with respect to the energy inner product $< \cdot, \cdot >_e$ and has compact resolvent. Consequently the eigenvectors of $\tilde{\mathcal{A}}$ are complete and orthogonal with respect to $< \cdot, \cdot >_e$. Furthermore the eigenvector estimates remain valid for the eigenvectors of $\tilde{\mathcal{A}}$. Therefore, due to the $\mathcal{O}(1)$ difference between $\lambda^\pm_{k,j}$ and $\tilde{\lambda}^\pm_{k,j}$, it is easy to show that $\sum_{k=1}^\infty \sum_{j=1,3} \sum_{+,-} ||\tilde{Y}^\pm_{j,k} - Y^\pm_{j,k}||^2 < \infty$. Hence by Bari's Theorem,[4] $\{Y_{\lambda^\pm_{k,j}}\}$ forms a Riesz basis for the orthogonal complement of E_0 if it can be established that they are ω-independent. Let $E(S)$ denote the span of the eigenvectors and generalized eigenvectors corresponding to $\lambda \in S$. It is easy to see that $E(S_k)$ is orthogonal to $E(S_j)$ for $j \neq k$. (where S_k is defined in Proposition 1). If $\sum_{k=1}^\infty \sum_{\lambda \in S_k} c_\lambda Y_\lambda = 0$, then by the block orthogonality we have $\sum_{\lambda \in S_k} c_\lambda Y_\lambda = 0, k = 0, 1, 2, 3, ...$ However for all blocks S_k, the eigenvectors and generalized eigenvectors are known to exist and be linearly independent (as they are derived from a matrix eigenvalue problem). Hence we obtain $c_\lambda = 0, \forall \lambda \in \sigma(\mathcal{A})$. Thus $\{Y_{\lambda^\pm_{k,j}}\}$ is an ω-linearly independent set. Proposition 1 is proved.

Proposition 2. *Let $\tilde{G}_2 > 0$. Assume the wave speeds $\sqrt{\frac{E_1}{\rho_1}}, \sqrt{\frac{E_3}{\rho_3}}$ are distinct, and*

$$\{\frac{K}{\alpha + \frac{m}{\sigma_k^2}}\}_{k=1}^\infty \bigcap \{\frac{E_1}{\rho_1}, \frac{E_3}{\rho_3}\} \neq \emptyset.$$

Then every non-zero eigenvalue has negative real part.

Proof: If $\{w, \mathbf{v}_\mathcal{O}\}$ is a smooth solution of the homogenous problem, then an easy calculation shows that

$$\frac{d\mathcal{E}}{dt} = -h_2\tilde{G}_2(\dot{\varphi}, \dot{\varphi})_\Omega$$

and hence for an eigenvalue to exist on the imaginary axis, $\dot{\varphi} = 0$ a.e. Since the eigenvectors satisfy (10), we must have $\lambda\varphi = 0$ in (26),(27), and since $\lambda \neq 0$ we have that $\varphi = 0$. This means eigenvectors satisfy (26),(27) with $G_2 = \tilde{G}_2 = 0$. However, such eigensolutions are readily calculated explicitly. One finds that the only way any of these solutions can have $\varphi = 0$ is if $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$. However this is not possible under the above hypothesis. This completes the proof.

Remark 2. If $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$ then there is an infinite family of eigenvectors corresponding to stretching motions as described in (21) that are completely undamped. Likewise if the condition given in Proposition 2 fails to be satisfied, it is possible to find appropriate physical parameters so that bending motions are possible that do not dissipate energy.

Using the fact that the eigenvectors form a Riesz basis, it is well known (see for e.g. Hansen[5]) that the growth of the semigroup is determined by the real part of the spectrum. Thus the following holds.

Corollary 1. *All finite energy solutions to the homogeneous problem (1)-(2) (with $M(t) = 0,\ \mathbf{g}_\mathcal{O}(t) = 0$) have energy $\mathcal{E}(t)$ that satisfies $\mathcal{E}(t) \leq Me^{-\gamma t}\mathcal{E}(0)$, where M is independent of initial data and $-\gamma$ is the supremum of the real parts of the eigenvalues of $\mathcal{A}$. Furthermore, $\gamma > 0$ if and only if the hypothesis of Proposition 2 is satisfied.*

4. SOLUTION OF THE MOMENT PROBLEM

We analyze the moment problem for the case of equal wave speeds (For the case of distinct wave speeds see Hansen and Rajaram[3]). Let us assume that the initial data given is zero, and determine which states are reachable in time T. We write the terminal state as

$$Y(T) = \sum_{\lambda\in\sigma(\mathcal{A})} c_\lambda Y_\lambda; \qquad c_\lambda =< Y(T), Y_\lambda^* >_e$$

where Y_λ^* is the eigenvector of $\mathcal{A}^*$ with eigenvalue $\bar{\lambda}$. The eigenvectors of $\mathcal{A}^*$ are the same as the eigenvectors of $\mathcal{A}$ except for the negative sign corresponding to any term that multiplies $\tilde{G}_2$. We need the following calculation

$$\begin{aligned}\left\langle\begin{pmatrix} 0 \\ \mathcal{B}\{M, \mathbf{g}_\mathcal{O}\}\end{pmatrix}, Y_\lambda^*\right\rangle_e &= c(\mathcal{B}\{M, \mathbf{g}_\mathcal{O}\}, V_\lambda^*) = c(\begin{pmatrix} J^{-1}M(t)\delta_L' \\ \mathbf{h}_\mathcal{O}^{-1}\mathbf{p}_\mathcal{O}^{-1}\mathbf{g}_\mathcal{O}\delta_L\end{pmatrix}, \begin{pmatrix} v_\lambda^* \\ \mathbf{v}_\lambda^*\end{pmatrix}) \\ &= < M(t)\delta_L', v_\lambda^* > + < \mathbf{g}_\mathcal{O}, \delta_L \mathbf{v}_\lambda^* > \\ &= \begin{cases} (-1)^k(g_1 + g_3) + \{M, \mathbf{g}_\mathcal{O}\}\vec{\mathcal{O}}(k^{-1}) & \text{if } j = 0 \\ (-1)^k[M + Nh_2/2(g_1 - g_3)] + \{M, \mathbf{g}_\mathcal{O}\}\vec{\mathcal{O}}(k^{-1}) & \text{if } j = 1 \\ (-1)^k[-Nh_2M + (1/h_2h_1)g_1 - (1/h_2h_3)g_3] + \{M, \mathbf{g}_\mathcal{O}\}\vec{\mathcal{O}}(k^{-1}) & \text{if } j = 3. \end{cases}\end{aligned}$$

where

$$\{M, \mathbf{g}_\mathcal{O}\} = (M(t), g_1(t), g_3(t)),$$

$$\vec{\mathcal{O}}(k^{-1}) = (\epsilon_{0,k}, \epsilon_{1,k}, \epsilon_{3,k})^T,\ \epsilon_{i,k} = \mathcal{O}(k^{-1}),\ j = 0, 1, 3.$$

For convenience, we define the controls $\{f_0, f_1, f_3\}$ so that the above (with $\lambda = \lambda_{k,j}^\pm$) becomes ,

$$b_\lambda = \left\langle\begin{pmatrix} 0 \\ \mathcal{B}\{M, \mathbf{g}_\mathcal{O}\}\end{pmatrix}, Y_\lambda^*\right\rangle_e = (-1)^k f_j + (f_0(t), f_1(t), f_3(t))\vec{\mathcal{O}}(k^{-1}). \tag{48}$$

Note that all the coefficients in b_λ that multiply the controls are bounded. As a consequence of the Carleson measure criterion (see for e.g. Ho and Russell[6] or Hansen and Weiss[7]), (8) is well posed on $X_0 \times X_1$ i.e. given any initial data in $X_0 \times X_1$, and L^2 controls $M(t), \mathbf{g}_\mathcal{O}(t)$, there exists a unique solution Y of (8) for which

$$Y = (U, V)^T \in C([0, T]; X_0 \times X_1). \tag{49}$$

The variation of parameters solution can be written

$$Y(T) = \int_0^T e^{\mathcal{A}(T-s)} \begin{pmatrix} 0 \\ \mathcal{B}\{M, \mathbf{g}_{\mathcal{O}}\} \end{pmatrix} ds = \int_0^T e^{\mathcal{A}t} \begin{pmatrix} 0 \\ \mathcal{B}\{\tilde{M}, \tilde{\mathbf{g}}_{\mathcal{O}}\} \end{pmatrix} ds \tag{50}$$

$$\text{where } \tilde{M}(t) = M(T-t), \quad \tilde{\mathbf{g}}_{\mathcal{O}} = \mathbf{g}_{\mathcal{O}}(T-t). \tag{51}$$

Hence multiplying the above by the eigenvectors Y_λ^* of $\mathcal{A}^*$ gives

$$c_\lambda = < Y(T), Y_\lambda^* >_e = < \int_0^T e^{\mathcal{A}t} \begin{pmatrix} 0 \\ \mathcal{B}\{\tilde{M}, \tilde{\mathbf{g}}_{\mathcal{O}}\} \end{pmatrix}, Y_\lambda^* \, dt >_e \tag{52}$$

$$= \int_0^T e^{\lambda t} < \begin{pmatrix} 0 \\ \mathcal{B}\{\tilde{M}, \tilde{\mathbf{g}}_{\mathcal{O}}\} \end{pmatrix}, Y_\lambda^* >_e dt \tag{53}$$

$$= \int_0^T e^{\lambda t} (\tilde{f}_j (-1)^k + (f_0, f_1, f_3) \vec{\mathcal{O}}(k^{-1})) \, dt. \tag{54}$$

For $k \in \mathbb{N}$, this results in the three moment problems

$$c_{0,k}^{\pm} = \int_0^T e^{\lambda_{0,k} t} (\tilde{f}_0(t) + \mathcal{O}(k^{-1}) \tilde{f}_1(t) + \mathcal{O}(k^{-1}) \tilde{f}_3(t)) \, dt \tag{55}$$

$$c_{1,k}^{\pm} = \int_0^T e^{\lambda_{1,k} t} (\mathcal{O}(k^{-1}) \tilde{f}_0(t) + \tilde{f}_1(t) + \mathcal{O}(k^{-1}) \tilde{f}_3(t)) \, dt \tag{56}$$

$$c_{3,k}^{\pm} = \int_0^T e^{\lambda_{3,k} t} (\mathcal{O}(k^{-1}) \tilde{f}_0(t) + \mathcal{O}(k^{-1}) \tilde{f}_1(t) + \tilde{f}_3(t)) \, dt. \tag{57}$$

The finite system corresponding to the nullvector (16), the generalized nullvector (17), and the eigenvectors described in (25) are

$$c_{0,0} = \int_0^T 0 \, dt \tag{58}$$

$$c_{0,1} = \int_0^T \tilde{g}_1 + \tilde{g}_3 \, dt \qquad c_{0,3}^{\pm} = \int_0^T e^{\lambda_{0,3}^{\pm} t} (h_3 \rho_3 \tilde{g}_1 - h_1 \rho_3 \tilde{g}_3) \, dt \tag{59}$$

Remark 3. As is easy to see from (58), it is not possible to steer a solution of (1) from the origin to the state corresponding to the null vector solution $w = 0$, $\mathbf{v}_{\mathcal{O}} = (1, 1)^T$.

Theorem 3. *Assume the wave speeds are distinct. Given any $\{c_\lambda\} \in \ell^2$ there exists functions $M(t)$ and $\mathbf{g}_{\mathcal{O}}(t)$ in $L^2(0, T)$ which solve the three moment problems (55)–(57), for all $k > K$, where K is sufficiently large in any time $T > \tau$, where the control time τ is given in (3).*

Proof: The proof is the same as the proof for the case when only (55) and (56) are present. Thus to give the idea we consider the following isomorphic coupled moment problem:

$$a_k = \int_0^T e^{s_k t} (f(t) + \delta_k g(t)) \, dt, \quad b_k = \int_0^T e^{\lambda_k t} (\epsilon_k f(t) + g(t)) \, dt \quad k \in \mathbb{Z}. \tag{60}$$

It is assumed that the sequences $\{s_k\}$ and $\{\lambda_k\}$ are each (individually) sequences of complex numbers lying in a vertical strip of $\mathbb{C}$ such that

$$\lim_{|k| \to \infty} (\mathrm{Im}\, s_k - Ak) = 0, \qquad \lim_{|k| \to \infty} (\mathrm{Im} \lambda_k - Bk) = 0. \tag{61}$$

Under these conditions, there is a uniform separation of the eigenvalues in each sequence for $|k| > K_0$ sufficiently large. Hence it is well-known that $\{e^{s_k t}\}_{|k|\geq K_0}$ forms a Riesz basis for its closure in $L^2(0,T)$, for any $T > 2\pi/A$. Similarly $\{e^{\lambda_k t}\}_{k\geq K}$ forms a Riesz basis for its closure in $L^2(0,T)$, for any $T > 2\pi/B$. For the moment, let us assume the separation condition holds. We first prove that if the sequences $\{\delta_k\}$ and $\{\epsilon_k\}$ have sufficiently small ℓ^2 norms, then there exists a solution to (60). Let $T > 2\pi/A$, $T > 2\pi/B$. Let $\{p_k\}$, $\{q_k\}$ be the biorthogonal sequences to $\{e^{s_k t}\}$,$\{e^{\lambda_k t}\}$. We seek f, g which solve

$$\begin{pmatrix} f \\ g \end{pmatrix} = K \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} \sum_{k\in\mathbb{Z}} \; [a_k - \delta_k < e^{s_k t}, g >]p_k(t)\,dt \\ \sum_{k\in\mathbb{Z}} \; [b_k - \epsilon_k < e^{\lambda_k t}, f >]q_k(t)\,dt \end{pmatrix}. \tag{62}$$

If such f, g exist, then it is a simple matter to see that (60) is satisfied. Since $\{\delta_k\}$ and $\{\epsilon_k\}$ are ℓ^2 sequences, and $\{e^{s_k t}\}$, $\{e^{\lambda_k t}\}$ form an $L - basis$ in $L^2(0,T)$, if follows that K maps $L^2(0,T)$ into itself. We show K is a contraction:

$$\begin{aligned}
\|K \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \quad &- \quad K \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \|^2_{L^2(0,T)} \\
&= \quad \| \sum_{k=1}^{\infty} \begin{pmatrix} -\delta_k < e^{s_k t}, g_1 - g_2 > p_k \\ -\epsilon_k < e^{\lambda_k t}, f_1 - f_2 > q_k(t) \end{pmatrix} \|^2_{L^2(0,T)} \\
&\leq \quad C(\|\{\delta_k < e^{s_k t}, g_2 - g_1 >\}\|^2_{\ell^2} + \|\{\epsilon_k < e^{\lambda_k t}, f_2 - f_1 >\}\|^2_{\ell^2}) \\
&\leq \quad C(\sum_{k=1}^{\infty} |\delta_k|^2| < e^{s_k t}, g_2 - g_1 > |^2 + |\epsilon_k|^2| < e^{\lambda_k t}, f_2 - f_1 > |^2 \\
&\leq \quad CT \sum_{k=1}^{\infty} |\delta_k|^2 \|g_2 - g_1\|^2 + |\epsilon_k|^2 \|f_2 - f_1\|^2 \\
&\leq \quad CT(\sum_{k=1}^{\infty} |\delta_k|^2 + |\epsilon_k|^2) \| \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} - \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \|^2_{L^2(0,T)}.
\end{aligned}$$

Thus if the ℓ^2 norms of ϵ_k and δ_k are small enough, K is a contraction. Since $\{\delta_k\}$ and $\{\epsilon_k\}$ are $\mathcal{O}(k^{-1})$, it follows that for N sufficiently large, we can find $f = f_N$ and $g = g_N$, both in $L^2(0,T)$, which solves (60) for $|k| \geq N$. Applying the same idea to (55)–(57) proves the theorem.

Remark 4. Keeping Remark 3 in mind, one can ask whether it is possible to solve (55)–(57) for all k together with (59). If possible, then exact controllability of (1)-(2) holds modulo the one dimensional uncontrollable quotient described in Remark 3. A sufficient condition for this result is that the eigenvalues grow (are not repeated) along each branch. It is not hard to show that this will hold if the coupling between the equations is sufficiently small and $\left\{ \dfrac{K\sigma_k^4}{m + \alpha\sigma_k^2} \right\}_{k=1}^{\infty}, \sigma = \dfrac{k\pi}{L}$ is a sequence of distinct numbers. Indeed for sufficiently small coupling, the l^2 norms of the coupling constants $\{\delta_k\}, \{\epsilon_k\}$ in the proof of Theorem 3 can be made arbitrarily small so that Theorem 3 is valid for the moment problems given by (55)–(57) for all k together with (59).

5. CONTROLLABILITY RESULTS

The fact that we can obtain $\{(f_0, f_1, f_3) \in (L^2(0,T))^3\}$ for $T > \tau$ which solves the moment problem for $k \geq K$, implies that a corresponding $\{M, \tilde{\mathbf{g}}_{\mathcal{O}}\}$ exists that "exactly controls" the high frequency portion of the state space. More precisely, let $\mathcal{P}_\infty$ denote the spectral projection operator defined on $X_1 \times X_0$ by

$$\mathcal{P}_\infty(\sum_{k=1}^{\infty} \sum_{\lambda\in S_k} c_\lambda Y_\lambda) = \sum_{k\geq K} \sum_{\lambda\in S_k} c_\lambda Y_\lambda,$$

where K is the integer defined in Theorem 3.

Theorem 4. *Given any initial data $Y_0 \in X_1 \times X_0$ and $T > \tau$ (τ as defined in (3)), there exists $\{M, \tilde{\mathbf{g}}_{\mathcal{O}}\} \in (L^2(0,T))^3$ such that the solution $Y(t)$ of (8) satisfies (49) and $\mathcal{P}_\infty Y(t) = 0$, $\forall t \geq T$.*

In view of Remark 4 in mind, we also have the following corollary.

Corollary 2. *If G_2 and $\tilde{G}_2$ are sufficiently small and* $\left\{\dfrac{K\sigma_k^4}{m+\alpha\sigma_k^2}\right\}_{k=1}^{\infty}, \sigma_k = \dfrac{k\pi}{L}$ *is a sequence of distinct numbers, then for $T > \tau$ Equation (1) is exactly controllable in the quotient space* $(X_0 \times X_1)/(0,1,1)^T$.

In the case where the wave speeds $\sqrt{\frac{E_1}{\rho_1}}, \sqrt{\frac{E_3}{\rho_3}}$ coincide, the Theorem 4 can be improved due to the fact that one of the equations of motion in (1) completely decouples. We make the following variable substitution.

$$z = \rho_1 h_1 v_1 + \rho_3 h_3 v_3, \qquad y = v_1 - v_3.$$

Then (1), (2) (with $\mu = \sqrt{\frac{E_1}{\rho_1}} = \sqrt{\frac{E_3}{\rho_3}}$) decouples into the following two systems:

$$\left\{\begin{array}{rcl} m\ddot{w} - \alpha D_x^2\ddot{w} + KD_x^4 w - D_x N h_2(G_2\varphi + \tilde{G}_2\dot{\varphi}) & = & 0 \quad \text{on } (0,L)\times(0,\infty) \\ \ddot{y} - \mu^2 D_x^2 y - (\frac{1}{\rho_1 h_1} + \frac{1}{\rho_3 h_3})(G_2\varphi + G_2\dot{\varphi}) = 0 & = & 0 \quad \text{on } (0,L)\times(0,\infty) \\ (\text{where } \varphi = h_2^{-1} y + N w_x) & & \\ w(0,t) = D_x^2 w(0,t) = D_x y(0,t) = w(L,t) & = & 0 \quad t > 0, \\ D_x^2 w(0,t) = M(t), D_x y(L,t) = F_1(t) & = & g_1(t) - g_3(t) \quad t > 0 \end{array}\right. \tag{63}$$

$$\left\{\begin{array}{rcl} \ddot{z} - \mu^2 D_x^2 z & = & 0 \\ D_x z(0,t) & = & 0,\ D_x z(L,t) = F_2(t) = \rho_1 h_1 g_1(t) + \rho_3 h_3 g_3(t). \end{array}\right. \tag{64}$$

It is routine to show that the z system (64) is exactly controllable modulo the "zero energy" solution $z \equiv$ constant. Furthermore the argument in the proof of Theorem 3 remains valid for the system (63),(64). Hence, we have the following remarks.

Remark 5. Under the hypothesis of Theorem 4 in the case of equal wave speeds ($\sqrt{E_1/\rho_1} = \sqrt{E_3/\rho_3}$), we can add the following
(i): there exists $u \in L^2(0,T)$ such that by using controls $M(t) \equiv 0$, $\mathbf{g}_{\mathcal{O}} = \{u(t), u(t)\}^T$ (1) satisfies

$$\rho_1 h_1 v_1 + \rho_3 h_3 v_3 = \text{ constant } \forall t \geq T,$$

(ii): there exists $u(t), M(t) \in L^2(0,T)$ such that the solution to (1),(2) with $\mathbf{g}_{\mathcal{O}}(t) = \{\frac{u(t)}{\rho_1 h_1}, \frac{u(t)}{\rho_3 h_3}\}^T$ satisfies

$$\{w, (v_1 - v_3)\} = 0 \,(\bmod Q)$$

where Q is a finite dimensional quotient defined in a manner similar to the quotient in Theorem 4
(iii): using controls $u(t), M(t) \in L^2(0,T)$ both (i) and (ii) above can be satisfied.

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