DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement Volume 2005

Website: http://AIMsciences.org

pp. **746–755**

NULL CONTROLLABILITY OF A DAMPED MEAD-MARKUS SANDWICH BEAM

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Abstract. The Mead-Markus sandwich beam model with shear damping is shown to be null controllable modulo a one dimensional state in an arbitrarily short time. The moment method is used to obtain this result.

1. **Introduction.** In this paper, boundary null-controllability of the Mead-Markus model [9] of a damped sandwich beam with shear damping is considered. The equations of motion as formulated in Fabiano and Hansen [1] are as given below :

$$mw_{tt} + (A + \frac{B^2}{C})w_{xxxx} - \frac{B}{C}s_{xxx} = 0 \tag{1}$$

$$\beta s_t + \gamma s - \frac{1}{C}s_{xx} + \frac{B}{C}w_{xxx} = 0, \tag{2}$$

with homogenous boundary conditions

$$w(0,t) = w(1,t) = 0, s_x(1,t) = 0, w_{xx}(1,t) = 0, \tag{3}$$

with controlled moment at the left end (see Hansen [6])

$$w_{xx}(0,t) = u(t), s_x(0,t) = Bu(t), \tag{4}$$

and initial conditions :

$$w(x,0) = w^0(x), w_t(x,0) = w^1(x), s(x,0) = s^0(x). \tag{5}$$

In the above, w denotes the transverse displacement of the beam, s is proportional to the shear of the middle layer, $u(t)$ represents moment control, m is the mass of the beam, A,B and C are material constants, γ and β are the elastic and damping coefficients of the middle layer respectively. For simplicity we assume that the beam is of unit length.

In this paper, we prove that (1)-(5) is null controllable in the sense that arbitrary initial states may be controlled to a particular one-dimensional state in which transverse displacement vanishes. The paper is organized as follows. In Section 2, the semigroup formulation of (1)-(5) is discussed. The spectral analysis of (1)-(5) is done in Section 3. The wellposedness of (1)-(5) is discussed in Section 4, and the moment problem and its solution is discussed in Section 5.

2000 *Mathematics Subject Classification.* Primary: 93B05, 80A20; Secondary: 70H25.
Key words and phrases. Sandwich beam, null controllability, boundary control .
The second author was supported in part by the NSF under grant DMS0205148.

2. **Semigroup formulation.** Let

$$\begin{pmatrix} w_1(t,x) \\ w_2(t,x) \\ w_3(t,x) \end{pmatrix} = \begin{pmatrix} w(t,x) \\ w_t(t,x) \\ s(t,x) \end{pmatrix}.$$

The arguments (t,x) will be omitted from now on for simplicity. First we consider the problem (1)-(5) with $u(t) = 0$ (i.e the case of homogenous boundary conditions) and obtain the following:

$$\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \mathcal{A}(\beta)\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \begin{pmatrix} w_1(0) \\ w_2(0) \\ w_3(0) \end{pmatrix} = \begin{pmatrix} w^0(x) \\ w^1(x) \\ s^0(x) \end{pmatrix},$$

where (using $D = \frac{\partial}{\partial x}$)

$$\begin{cases} \mathcal{A}(\beta) = \begin{pmatrix} 0 & I & 0 \\ -\frac{1}{m}(A + \frac{B^2}{C})D^4 & 0 & \frac{B}{Cm}D^3 \\ -\frac{B}{C\beta}D^3 & 0 & (-\frac{\gamma}{\beta} + \frac{1}{C\beta}D^2) \end{pmatrix}, \\ \\ \mathcal{D}(\mathcal{A}(\beta)) = H_w^4 \times H_w^2 \times H_s^3 \end{cases}$$

and

$$\begin{aligned} H_w^4 &= \{\phi \in H^4(0,1) : \phi(0) = \phi(1) = \phi_{xx}(0) = \phi_{xx}(1) = 0\} \\ H_w^2 &= \{\phi \in H^2(0,1) : \phi(0) = \phi(1) = 0\} \\ H_s^3 &= \{\phi \in H^3(0,1) : \phi_x(0) = \phi_x(1) = 0\}, \end{aligned}$$

$\mathcal{A}(\beta) : \mathcal{D}(\mathcal{A}(\beta)) \to \mathcal{H} = H_w^2 \times L^2(0,1) \times H^1(0,1)$ where $\mathcal{H}$ is a Hilbert space with the following energy inner product (see Fabiano, Hansen [1]) :

$$\left\langle \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\rangle_e = \frac{1}{2}\int_0^1 (g\bar{v} + \frac{A}{m} f_{xx}\bar{u}_{xx} + \frac{\gamma}{m} h\bar{w} + \frac{1}{Cm}(Bf_{xx} - h_x)(B\bar{u}_{xx} - \bar{w}_x)\, dx.$$

Theorem 1. *$\mathcal{A}(\beta)$ is the infinitesimal generator of a strongly continuous semigroup on $\mathcal{H}$. Furthermore $\mathcal{A}(\beta)$ extends by duality to an isomorphic semigroup on $\mathcal{H}_{-\frac{1}{2}} := H_0^1(0,1) \times H^{-1}(0,1) \times L^2(0,1)$.*

Proof. We can easily check that $\mathcal{A}(\beta)^* = -(\mathcal{A}(-\beta))$ and $D(\mathcal{A}(\beta)^*) = D(\mathcal{A}(\beta))$ by direct calculation. Next we check that both $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative on $\mathcal{H}$. Then the result will follow from the Lumer-Phillips theorem (see for e.g. Pazy [10]). We show the calculation for $\mathcal{A}$. The calculation for $\mathcal{A}^*$ is similar.

$$\begin{aligned} \left\langle \mathcal{A}\begin{pmatrix} f \\ g \\ h \end{pmatrix}, \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\rangle_e &= \frac{1}{2}\int_0^1 \frac{-1}{m}(A + \frac{B^2}{C}) f_{xxxx}\bar{g} + \frac{B}{Cm} h_{xxx}\bar{g} + \frac{A}{m} g_{xx}\bar{f}_{xx} \\ &+ \frac{\gamma}{m}(\frac{-B}{C\beta} f_{xxx}\bar{h} - \frac{\gamma}{\beta} h\bar{h} + \frac{1}{C\beta} h_{xx}\bar{h}) \\ &+ \frac{1}{Cm}(Bg_{xx} + \frac{B}{C\beta} f_{xxxx} + \frac{\gamma}{\beta} h_x - \frac{1}{C\beta} h_{xxx})(B\bar{f}_{xx} - \bar{h}_x) \end{aligned}$$

$$= -\frac{\gamma^2}{m\beta}|h|^2 - \frac{B^2}{mC^2\beta}|f_{xxx}|^2 - \frac{B\gamma}{mC\beta}f_{xxx}\bar{h} - \frac{B\gamma}{mC\beta}\bar{f}_{xxx}h$$
$$= \frac{-1}{2m\beta}|\gamma h + \frac{B}{C}f_{xxx}|^2 \leq 0.$$

The semigroup is easily extended by duality to $\mathcal{H}_{-1} = (\mathcal{D}(\mathcal{A}(\beta)^*))^*$ as in e.g., [12] The interpolation to $\mathcal{H}_{-\frac{1}{2}} := H_0^1(0,1) \times H^{-1}(0,1) \times L^2(0,1)$ is easily justified once the Riesz basis property is proved. Hence pending the proof of Theorem 2, Theorem 1 is proved. □

Using a standard integration by parts against a function in $\mathcal{D}(\mathcal{A}(\beta))$, one can reformulate (1) - (5) as a problem with homogenous boundary conditions, but non-homogenous right hand side. Formally, one obtains the following system:

$$\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \mathcal{A}(\beta)\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + fu(t) \tag{6}$$

$$f = \begin{pmatrix} 0 \\ \frac{1}{m}(A + \frac{B^2}{C})\delta' \\ 0 \end{pmatrix}, \quad \begin{pmatrix} w_1(0) \\ w_2(0) \\ w_3(0) \end{pmatrix} = \begin{pmatrix} w^0(x) \\ w^1(x) \\ s^0(x) \end{pmatrix}, \tag{7}$$

where in (7), δ' is the distributional derivative of the Dirac delta distribution. Henceforth we adopt the above formulation of (1) - (5). We will see later that (6) - (7) is well posed on a subspace of $\mathcal{H}_{-\frac{1}{2}}$.

3. **Spectral Analysis of $\{\mathcal{A}(\beta)\}$.** Under some parametric restrictions, it is shown in Lemma 1 that the spectrum of $\mathcal{A}(\beta)$ (henceforth denoted by Λ) consists of a negative real branch $\{\lambda_{k,1}\}_{k=1}^{\infty} \bigcup \lambda_0$ and two complex conjugate branches $\{\lambda_{k,2}\}_{k=1}^{\infty}$, $\{\lambda_{k,3}\}_{k=1}^{\infty}$, where $\lambda_0 = -\frac{\gamma}{\beta}$. It is also shown that each of the three branches grow asymptotically at a quadratic rate with $\lim_{k\to\infty} \arg(-\lambda_{k,j}) = \theta$, for some $\theta \in (0, \frac{\pi}{2}), j = 1, 2, 3$. The eigenfunctions can be calculated and are given by the following:

$$\phi_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \{\phi_{k,j}\} = \begin{pmatrix} \frac{1}{\lambda_{k,j}}\sin(\alpha_k x) \\ \sin(\alpha_k x) \\ A_{k,j}\cos(\alpha_k x) \end{pmatrix}, \tag{8}$$

$$A_{k,j} = \frac{\lambda_{k,j} + \frac{1}{m}(A + \frac{B^2}{C})\frac{\alpha_k^4}{\lambda_{k,j}}}{\frac{B}{C}\alpha_k^3}, \tag{9}$$

where $\alpha_k = k\pi$ and $\lambda_{k,j}$'s are the solutions of the following cubic equation [1] :

$$\beta m\lambda_k^3 + (\frac{\alpha_k^2}{C} + \gamma)m\lambda_k^2 + \beta\frac{AC + B^2}{C}\alpha_k^4\lambda_k + \frac{AC + B^2}{C}\alpha_k^4(\gamma + \frac{A}{AC + B^2}\alpha_k^2) = 0. \tag{10}$$

The following change of variables

$$x = \sqrt{\frac{m}{A}}\frac{1}{\alpha_k^2}\lambda_k, \ \bar{\beta} = \sqrt{\frac{m}{A}}\frac{1}{C\beta}, \ \kappa = 1 + \frac{B^2}{AC},$$

converts (10) to the following:

$$x^3 + \bar{\beta}x^2 + \kappa x + \bar{\beta} + \frac{\bar{\beta}C\gamma}{\alpha_k^2}(x^2 + \kappa) = 0. \tag{11}$$

Lemma 1. *If $\kappa \in (1, 9], \bar{\beta} > \frac{\kappa}{\sqrt{2}}$, and $\gamma < \frac{1}{\kappa C}$ the eigenvalues are separated and the following estimates hold for $j = 1, 2, 3$:*

$$\exists \quad \theta \in (0, \frac{\pi}{2}) \text{ such that } |arg(-\lambda_{k,j})| \leq \theta, \ \forall \ k \in \mathbb{N}, \tag{12}$$

$$\exists \quad C_1, C_2 > 0 \text{ such that } C_1 k^2 \leq |\lambda_{k,j}| \leq C_2 k^2, \ \forall \ k \in \mathbb{N}, \tag{13}$$

$$\exists \quad \bar{\sigma} > 0 \text{ such that } |\lambda_{m,j} - \lambda_{n,j}| \geq \bar{\sigma}|m^2 - n^2|, \ \forall \ m, n \in \mathbb{N}. \tag{14}$$

Proof. We observe that as β takes all values from 0 to ∞, so does $\bar{\beta}$ and vice-versa. We also observe that as $k \to \infty$, Rouche's theorem implies that the roots of (11) with $\gamma \neq 0$ will be close to the roots of (11) with $\gamma = 0$. Thus, let us consider the case $\gamma = 0$ first. Then we are interested in the roots of

$$f(x) = x^3 + \bar{\beta}x^2 + \kappa x + \bar{\beta} = 0. \tag{15}$$

It can be shown [1] that if $1 < \kappa \leq 9$ then (15) has one negative real root (say $a < 0$) and a pair of complex conjugate roots ($b \pm ci, \ b < 0$). Each root of (10) with $\gamma = 0$ is a real multiple of one of the three roots of (15). The roots of (10) with $\gamma = 0$ are given by

$$s_{k,1} = a(k\pi)^2\sqrt{\frac{A}{m}}, \ s_{k,2} = (b + ci)(k\pi)^2\sqrt{\frac{A}{m}}, \ s_{k,3} = (b - ci)(k\pi)^2\sqrt{\frac{A}{m}}.$$

The argument of the branches is given by the argument of the roots of (15). Hence the roots lie within a sector in the left half plane; i.e. $\exists \ \theta \in (0, \frac{\pi}{2})$ such that $|\arg(-s_{k,j})| \leq \theta \ \forall \ k \in \mathbb{N}, j = 1, 2, 3$. The $s_{k,j}$'s satisfy $|s_{m,j} - s_{n,j}| \geq \sigma|(m^2 - n^2)|, \ \forall \ m, n \in \mathbb{N}$ for some $\sigma > 0$. Now, we return to the roots of (10) with $\gamma \neq 0$. The following change of variables

$$y = x\sqrt{\frac{1 + \frac{C\gamma}{\alpha_k^2}}{1 + \frac{C\gamma\kappa}{\alpha_k^2}}}$$

converts (11) to the following form:

$$y^3 + \bar{\beta}'y^2 + \kappa' y + \bar{\beta}' = 0, \tag{16}$$

where

$$\bar{\beta}' = \bar{\beta}\frac{(1 + \frac{C\gamma}{\alpha_k^2})^{\frac{3}{2}}}{\sqrt{1 + \frac{C\gamma\kappa}{\alpha_k^2}}}, \ \kappa' = \kappa\sqrt{\frac{1 + \frac{C\gamma}{\alpha_k^2}}{1 + \frac{C\gamma\kappa}{\alpha_k^2}}}.$$

Hence (16) can be handled in a way similar to (15) to conclude that if $1 < \kappa \leq 9$, then (11) has one negative real root ($a < 0$) and a pair of complex conjugate roots ($b \pm ci, \ b < 0$) with negative real part $\forall \ k \in \mathbb{N}$ and $\exists \ C_1, C_2 > 0$ such that $C_1 k^2 \leq |\lambda_{k,j}| \leq C_2 k^2, \ \forall \ j = 1, 2, 3, k \in \mathbb{N}$. Hence (13) holds.

Next we show that $\forall \ \bar{\beta} > \frac{\kappa}{\sqrt{2}}, \ \lambda_{k_1,j} \neq \lambda_{k_2,j}$ if $k_1 \neq k_2$. Let $\rho = \sqrt{\frac{A}{m}}, \epsilon = \frac{C\gamma}{k^2\pi^2}$. Let λ denote an eigenvalue which is common to $k = k_1, k_2$. Then rewriting (10) in terms of λ we get the following:

$$\lambda^3 + \rho k^2 \bar{\beta}(1 + \epsilon)\lambda^2 + \rho^2 k^4 \kappa\lambda + \bar{\beta}(1 + \epsilon\kappa)\rho^3 k^6 = 0. \tag{17}$$

Subtracting the equations corresponding to $k = k_1$ and $k = k_2$ in (17), we get

$$\lambda^2\rho\bar{\beta}(k_1^2 - k_2^2) + \rho^2\kappa(k_1^4 - k_2^4)\lambda + \bar{\beta}\rho^3(k_1^6 - k_2^6) + \bar{\beta}\frac{C\gamma\kappa}{\pi^2}\rho^3(k_1^4 - k_2^4) = 0 \quad (18)$$

$$(k_1^2 - k_2^2)\left[\lambda^2\bar{\beta} + \rho\kappa(k_1^2 + k_2^2)\lambda + \kappa\rho^2(k_1^4 + k_2^4 + k_1^2k_2^2) + \bar{\beta}\frac{C\gamma\kappa}{\pi^2}\rho^2(k_1^2 + k_2^2)\right] = 0. \quad (19)$$

It can be checked that if $\bar{\beta} > \frac{\kappa}{\sqrt{2}}$ then the quadratic in (19) has no real roots and hence there is no common real root between $k = k_1$ and $k = k_2$. Using the fact that the real roots of (17) are distinct, one can show the same for the complex conjugate roots. Hence the eigenvalues $\{\lambda_{k,j}\}$ are distinct for all $k \in \mathbb{N}$ and $j = 1, 2, 3$.

Next we obtain estimates on $\lambda_{k,j}$'s. We choose $k \geq K_0$ such that $|\lambda_{k,j} - s_{k,j}| < \epsilon_0 = \frac{\sigma}{4}$, $\forall j = 1, 2, 3$. Then $\forall\ m, n \in \mathbb{N}$,$m, n \geq K_0$ and $j = 1, 2, 3$, we have

$$\begin{aligned}
|\lambda_{m,j} - \lambda_{n,j}| &\geq |s_{m,j} - s_{n,j}| - |\lambda_{m,j} - s_{m,j}| - |\lambda_{n,j} - s_{n,j}| \\
&\geq \sigma|m^2 - n^2| - 2\epsilon_0 \\
&= \sigma|m^2 - n^2| - \frac{\sigma}{2} \\
&\geq \frac{\sigma}{2}|m^2 - n^2|.
\end{aligned}$$

Also $\forall\ m, n < K_0$ we have finitely many roots. Let $\hat{\sigma} = \min(|\lambda_{m,j} - \lambda_{n,j}| : m, n = 1, ..., K_0)$. Then

$$|\lambda_{m,j} - \lambda_{n,j}| \geq \hat{\sigma} \geq \frac{\hat{\sigma}}{K_0^2 - 1}|m^2 - n^2|\ \forall\ m, n < K_0.$$

Now let $\bar{\sigma} = min(\sigma, \frac{\hat{\sigma}}{K_0^2 - 1})$. Then we have

$$|\lambda_{m,j} - \lambda_{n,j}| \geq \bar{\sigma}|m^2 - n^2|,\ \forall\ m, n \in \mathbb{N}, j = 1, 2, 3.$$

Hence (14) holds. Also by a similar argument we can show that $\exists\ \theta \in (0, \frac{\pi}{2})$ such that $|\arg(-\lambda_{k,j})| \leq \theta$, $\forall k \in \mathbb{N}, j = 1, 2, 3$. Hence (12) holds. Finally, it can be shown that if $\gamma < \frac{1}{\kappa C}$ then $\lambda_0 \neq \lambda_{k,j}$, $\forall\ k \in \mathbb{N}, j = 1, 2, 3$ and this completes the proof of Lemma 1. □

Theorem 2. *$\{\phi_{k,j}\} \cup \{\phi_0\}$ forms a Riesz basis for $(\mathcal{H}, < . >_e)$.*

Proof. Let

$$\theta_0 = (0, 0, 1)^T, \theta_{k,1} = (\frac{\sqrt{2}}{k^2\pi^2}\sin(k\pi x), 0, 0)^T$$

$$\theta_{k,2} = (0, \sqrt{2}\sin(k\pi x), 0)^T, \theta_{k,3} = (0, 0, \frac{\sqrt{2}}{k\pi}\cos(k\pi))^T.$$

Then θ_k's are related to ϕ_k's in the following way:

$$\begin{aligned}
\phi_0 &= \theta_0 \\
\phi_{k,j} &= \frac{k^2\pi^2}{\sqrt{2}\lambda_{k,j}}\theta_{k,1} + \frac{1}{\sqrt{2}}\theta_{k,2} + \frac{k\pi A_{k,j}}{\sqrt{2}}\ \theta_{k,3},\ \forall\ k \in \mathbb{N}\ , j = 1, 2, 3.
\end{aligned}$$

It can be checked that $\{\theta_0\} \cup \{\theta_{k,j}\}$ is an orthogonal basis in $(\mathcal{H}, < . >_e)$. Also $\exists M > 0$ such that $\frac{1}{M} < ||\theta_{k,j}||_e < M$, $\forall k \in \mathbb{N}, j = 1, 2, 3$. Hence $\{\theta_0\} \cup \{\theta_{k,j}\}$ is equivalent to an orthonormal basis in the energy innerproduct. We define a mapping $L : \mathcal{H} \to \mathcal{H}$ as follows:

$$\begin{aligned} L(\theta_0) &= \phi_0 \\ L\left[\begin{pmatrix} \theta_{k,1} \\ \theta_{k,2} \\ \theta_{k,3} \end{pmatrix}\right] &= \begin{pmatrix} \phi_{k,1} \\ \phi_{k,2} \\ \phi_{k,3} \end{pmatrix}. \end{aligned}$$

Let

$$L_k = \begin{pmatrix} \frac{k^2\pi^2}{\lambda_{k,1}} & \frac{1}{\sqrt{2}} & \frac{k\pi A_{k,1}}{\sqrt{2}} \\ \frac{k^2\pi^2}{\lambda_{k,2}} & \frac{1}{\sqrt{2}} & \frac{k\pi A_{k,2}}{\sqrt{2}} \\ \frac{k^2\pi^2}{\lambda_{k,3}} & \frac{1}{\sqrt{2}} & \frac{k\pi A_{k,3}}{\sqrt{2}} \end{pmatrix}$$

denote the matrix of transformation of T between blocks. The set $\{\phi_{k,j}\}$ is block orthogonal with block size 3 corresponding to $j = 1, 2, 3$. Hence $\{\phi_{k,j}\} \cup \{\phi_0\}$ will be a Riesz basis if each of the block matrices L_k are uniformly bounded and invertible; see [2]. Using (13) in Lemma (1), it can be easily checked that $A_{k,j} \sim \mathcal{O}(\frac{1}{k})$, where $A_{k,j}$'s are described in (9). Hence L_k has an invertible limiting form for k large enough. Also L_k is invertible for small k due to the separation of the eigenvalues. Hence we also have that $|\det(L_k)| \leq C$ for some $C > 0$. Thus, $\{\phi_{k,j}\} \cup \{\phi_0\}$ forms a Riesz basis for $(\mathcal{H}, <.>_e)$. This proves Theorem 2. □

4. **Admissibility of input element.** Let J denote the set of indices for the eigenvalues of $\mathcal{A}(\beta)$. Let $\{\psi_k\}$ be a sequence biorthogonal to $\{\phi_k\}$ which are also the eigenvectors of $\mathcal{A}(\beta)^*$ and

$$\tilde{w}(t, x) = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \sum_{k \in J} w_k(t)\phi_k(x) \tag{20}$$

be the solution of (6) where ϕ_k's are the eigenvectors of $\mathcal{A}$ found above and $w_k(t)$'s are scalar functions of time. Also let

$$\tilde{w}(0, x) = \begin{pmatrix} w^0(x) \\ w^1(x) \\ s^0(x) \end{pmatrix} = \sum_{k \in J} w_k(0)\phi_k(x) \tag{21}$$

Substituting (20) in (6) , we get

$$\tilde{w}_t = \mathcal{A}\tilde{w} + \begin{pmatrix} 0 \\ \frac{1}{m}(A + \frac{B^2}{C})\delta' \\ 0 \end{pmatrix} u(t),$$

$$\Rightarrow \langle \tilde{w}_t, \psi_k(x) \rangle_e = \langle \mathcal{A}\tilde{w}, \psi_k(x) \rangle_e + \left\langle \begin{pmatrix} 0 \\ \frac{1}{m}(A + \frac{B^2}{C})\delta' \\ 0 \end{pmatrix} u(t), \psi_k(x) \right\rangle_e ,$$

$$\Rightarrow \left\langle \sum_{k \in J} w_k'(t)\phi_k(x), \psi_k(x) \right\rangle_e = \left\langle \sum_{k \in J} w_k(t)(\mathcal{A}\phi_k(x)), \psi_k(x) \right\rangle_e$$

$$+ \left\langle \sum_{k \in J} f_k\phi_k(x), \psi_k(x) \right\rangle_e u(t),$$

$$\Rightarrow w_k'(t) = \lambda_k w_k(t) + f_k u(t),$$

$$\Rightarrow w_k(t) = e^{\lambda_k t} w_k(0) + \int_0^t e^{\lambda_k(t-s)} f_k u(s) ds, \tag{22}$$

where

$$\begin{pmatrix} 0 \\ \frac{1}{m}(A + \frac{B^2}{C})\delta' \\ 0 \end{pmatrix} = \sum_{k \in J} f_k \phi_k(x),\ f_k = \left\langle \begin{pmatrix} 0 \\ \frac{1}{m}(A + \frac{B^2}{C})\delta' \\ 0 \end{pmatrix}, \psi_k(x) \right\rangle_e . \tag{23}$$

The ψ_k's can be computed explicitly and are as follows:

$$\psi_{k,j} = \begin{pmatrix} \frac{-1}{\hat{\lambda}_{k,j}} \sin(k\pi x) \\ \sin(k\pi x) \\ \hat{A_{k,j}} \cos(k\pi x) \end{pmatrix}, \psi_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where

$$\hat{\lambda}_{k,j} = \bar{\lambda}_{k,j},$$

$$\hat{A}_{k,j} = \frac{-\hat{\lambda}_{k,j} + \frac{1}{m}(A + \frac{B^2}{C})\frac{(k\pi)^4}{\hat{\lambda}_{k,j}}}{\frac{B}{Cm}(k\pi)^3}, k \in \mathbb{N}, j = 1, 2, 3. \tag{24}$$

The f_k's can also be computed and are given by the following:

$$f_0 = 0$$

$$f_{k,j} = -\frac{1}{m}(A + \frac{B^2}{C})k\pi,\ \forall k \in \mathbb{N}, j = 1, 2, 3. \tag{25}$$

Next we show that $\{f_{k,j}\}$ defined in (25) forms an admissible input for (22) in the Hilbert space $\mathcal{H}_{-\frac{1}{4}} = (\{d_k\} : \{\frac{d_k}{\sqrt[4]{|\lambda_k|}}\} \in l^2(J))$, where the norm is given by $||\{d_k\}||_{\mathcal{H}_{-\frac{1}{4}}} = ||\frac{d_k}{\sqrt[4]{|\lambda_k|}}||_{l^2(J)}$. We define $\mathcal{A}^{-\frac{1}{4}} : \mathcal{H}_{-\frac{1}{4}} \to l^2(J)$ in the following way:

$$\mathcal{A}^{-\frac{1}{4}}\{c_k\} = \{\frac{c_k}{\sqrt[4]{|\lambda_k|}}\}.$$

Then $\mathcal{A}^{-\frac{1}{4}}$ is an isomorphism from $\mathcal{H}_{-\frac{1}{4}}$ into $l^2(J)$.

Theorem 3. *$\{f_k\}_{k\in J}$ given by (25) forms an admissible input for (22) in $\mathcal{H}_{-\frac{1}{4}}$.*

Remark 1. Equivalently, if $\{w_k(0)\} \in \mathcal{H}_{-\frac{1}{4}}$, then $\{w_k(t)\}$ in (22) satisfies the following:

$$||\{w_k(t)\}||_{\mathcal{H}_{-\frac{1}{4}}} \le C_T(||\{w_k(0)\}||_{\mathcal{H}_{-\frac{1}{4}}} + ||u||_{L^2(0,T)}).$$

Proof. Equivalently, we prove that $\mathcal{A}^{-\frac{1}{4}}\{f_k\}$ forms an admissible input for $\mathcal{H}$. This will prove the theorem since $\mathcal{A}^{-\frac{1}{4}} : \mathcal{H}_{-\frac{1}{4}} \to l^2(J)$ is an isomorphism. We verify that $\{\mathcal{A}^{-\frac{1}{4}} f_k\}_{k\in J}$ satisfies the Carleson measure criterion, [3, 7]. Recall from (25) and Lemma 1 that $f_k \sim \mathcal{O}(k)$ and $\lambda_k \sim \mathcal{O}(k^2)$. Let $g_k = \mathcal{A}^{-\frac{1}{4}} f_k \sim \mathcal{O}(\sqrt{k})$. Then (22) can be rewritten in the following way.

$$\mathcal{A}^{-\frac{1}{4}} w_k(t) = \mathcal{A}^{-\frac{1}{4}} e^{\lambda_k t} w_k(0) + \int_0^t e^{\lambda_k(t-s)} g_k u(s) ds,\ \forall\ k \in J. \tag{26}$$

If we choose $\{w_k\} \in \mathcal{H}_{-\frac{1}{4}}$, then $\mathcal{A}^{-\frac{1}{4}}\{w_k\} \in l^2(J)$. We define the following rectangle in the complex plane.

$$R(h, \omega) = \{z \in C : 0 \le Re(z) \le h, |Im(z) - \omega| \le h\}.$$

Then we have,

$$\sum_{-\lambda_k \in R(k^2,\omega)} |g_k|^2 \leq \sum_{-\lambda_k \in R(k^2,0)} |g_k|^2 \leq \mathcal{O}(k^2), \ \forall \ h > 0, \tag{27}$$

where in (27) we have made use of the fact that the number of eigenvalues in $R(k^2, 0)$ is $\mathcal{O}(k)$ and that the worst case scenario happens when the rectangle is centered at the origin. Hence it follows that $\exists M > 0$ such that

$$\sum_{-\lambda_k \in R(h,\omega)} |g_k|^2 \leq Mh$$

and this proves Theorem 3. □

Remark 2. As a consequence, if the initial data in (7) belong to $\mathcal{H}_{-\frac{1}{4}}$ and $u(t) \in L^2(0,T)$ then there exists a unique solution $\{w, w_t, s\}$ to (6) - (7) defined by (20) and (22). Furthermore for some $C > 0$ we have the following:

$$||\{w, w_t, s\}(.,T)||_{\mathcal{H}_{-\frac{1}{4}}} \leq C(||\{w^0, w^1, s^0\}||_{\mathcal{H}_{-\frac{1}{4}}} + ||u||_{L^2(0,T)}).$$

5. **The moment problem and its solution.** For controllability to the zero state we seek $u(t) \in L^2(0,T)$ which solves (26) with $\{w_k(T)\} = 0, \ \forall k \in J$. Let $d_k = \mathcal{A}^{-\frac{1}{4}} w_k(0)$. Then we have $d_k \in l^2(J)$, and (26) can be rewritten as:

$$-\, d_0 \ = \ \int_0^T e^{\lambda_0 (t-s)} g_0 u(s) ds, \tag{28}$$

$$-d_{k,j} \ = \ \int_0^T e^{\lambda_{k,j}(t-s)} g_{k,j} u(s) ds, \ \forall k \in \mathbb{N}, j = 1, 2, 3. \tag{29}$$

Remark 3. In (28), $g_0 = \mathcal{A}^{-\frac{1}{4}} f_0 = 0$ and hence the eigenspace $\theta_0 = (0,0,1)^T$ cannot be controlled to the zero state. Hence we consider the solvability of (29).

From (25) we have that $\{f_{k,j}\}; k \in N, j = 1, 2, 3$ is bounded away from zero and hence we can rewrite (26) as follows:

$$\int_0^T e^{\lambda_{k,j} \tau} \tilde{u}(\tau) d\tau = c_{k,j}, \ \forall \ k \in \mathbb{N}, j = 1, 2, 3, \tag{30}$$

where

$$c_{k,j} = \frac{-\mathcal{A}^{-\frac{1}{4}} \{e^{\lambda_{k,j} T} w_{k,j}(0)\}}{\mathcal{A}^{-\frac{1}{4}} f_{k,j}}, \ \forall \ k \in \mathbb{N}, j = 1, 2, 3,$$

where T is the final time instant and $\tilde{u}(t) = u(T - t)$. Using Lemma 1 and (25) it can be shown that $\exists \alpha, B > 0$ such that $|c_{k,j}| \leq Be^{-\alpha k^2}, \ \forall \ k \in \mathbb{N}, j = 1, 2, 3$. If we are looking for a control in $L^2(0,T)$, then equation (30) can be rewritten as

$$\left\langle \tilde{u}(t), e^{\lambda_{k,j} t} \right\rangle_{L^2(0,T)} = c_{k,j}, \ \forall \ k \in \mathbb{N}, j = 1, 2, 3, \tag{31}$$

where $\{c_{k,j}\} \in l^2(J - \{0\})$. Hence the original problem has been transformed into the moment problem given by (31).

In order to solve the moment problem given by (31) we need the following theorem from Hansen [5] :

Theorem 4. *Let $\Lambda_0 := \{\lambda_k\}_{k=1}^{\infty}$ be a sequence of distinct complex numbers lying in $\Delta_\theta := \{\lambda \in \boldsymbol{C} \colon |arg(\lambda) \leq \theta\}$, which satisfy*

$$|\lambda_k - \lambda_j| \geq \rho |k^\beta - j^\beta|, (\beta > 1, \rho \geq 0),\ \epsilon(A + Bk^\beta) \leq \lambda_k < A + Bk^\beta.$$

Then there exist a sequence of functions $q_k(T,t)$ which are biorthogonal to $\{e^{\lambda_k t}\}$ in $L^2(0,T)$ and satisfy

$$e^{mk} < ||q_k(T,t)||_{L^2(0,T)} \leq K_T e^{kM}, (m, M > 0).$$

After reindexing the eigenvalues, it is clear from Lemma 1 that the eigenvalues $\{\lambda_k\}_{k\in J-\{0\}}$ satisfy the estimates needed by Theorem (4). Hence a solution to the moment problem given by (31) is given by

$$\tilde{u}(t) = \sum_{k\in J-\{0\}} c_k q_k(T,t).$$

We also have the following:

$$||\tilde{u}||_{L^2(0,T)} \leq \sum_{k\in J-\{0\}} |c_k| ||q_k||_{L^2(0,T)} \leq \sum_{k\in J-\{0\}} BK_T e^{-\alpha k^2} e^{kM} < \infty.$$

We conclude by stating the main theorem of the paper.

Theorem 5. *Assume $B < 2\sqrt{2AC}, \beta < \sqrt{\frac{2m}{A}\frac{1}{B^2 + AC}}$, and $\gamma < \frac{1}{\kappa C}$. Given any initial state $\{w^0, w^1, s^0\} \in \mathcal{H}_{-\frac{1}{4}}$ and $T > 0$, $\exists u \in L^2(0,\infty)$ supported on $(0,T)$ such that $\forall t \geq T$, $\{w, w_t, s\}(.,T) = \{(0, 0, e^{-\frac{\gamma}{\beta}T} K_0)\}$, where K_0 is a constant determined by the initial data.*

Remark 4. The restriction on β prevents the eigenvalues from being overdamped (asymptotically). Without the other parametric restrictions there is at most a finite number of repeated eigenvalues which could result in a lack of controllability.

Remark 5. The undamped model has been proved to be exactly controllable in $H_0^1(0,1) \times H^{-1}(0,1)$ [11]. Here in the damped case, we obtain greater regularity due to analytic smoothing (see Hansen and Lasiecka [4]). Thus, the control space is smaller. On the other hand, the homogenous problem is well posed in $\mathcal{H}_{-\frac{1}{2}} = H_0^1(0,1) \times H^{-1}(0,1) \times L^2(0,1)$. If we are given initial data in $H_0^1(0,1) \times H^{-1}(0,1) \times L^2(0,1)$ the zero control can be applied for a short time ϵ after which Theorem 5 applies. It is also interesting to note that the uncontrollable subspace does not exist in the undamped case.

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Received September, 2004; revised March, 2005.

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