HW 2

1a $\lim_{n\to\infty} \frac{n+1}{n-1/2} = 1$. Note that $\left|\frac{n+1}{n-1/2} - 1\right| = \left|\frac{n+1-(n+1/2)}{n-1/2}\right|$

$$= \frac{3/2}{n-1/2} \le \frac{3/2}{n-n/2}$$

$$= \frac{3}{n}$$

Let $\varepsilon > 0$. Pick $M \in \mathbb{N}$: $M > \frac{3}{\varepsilon}$.

Then if $n \ge M$,

$$\left|\frac{n+1}{n-1/2} - 1\right| \le \frac{3}{n} \le \frac{3}{M} < \varepsilon \quad \square$$

b Note that $\frac{3^n}{n!} = \frac{3\cdot 3\cdot 3\cdots 3}{1\cdot 2\cdot 3\cdots n} = \underbrace{\left(\frac{3\cdot 3}{1\cdot 2}\right)}_{9/2}\cdot\underbrace{\left(\frac{3}{3}\cdot\frac{3}{4}\cdots\frac{3}{n-1}\right)}_{\le 1}\cdot\frac{3}{n} \le \frac{27}{2n}$

valid if $n \ge 3$

Let $\varepsilon > 0$. Pick $N \in \mathbb{N}$ such that

$$N > \max\left\{3, \frac{27}{2\varepsilon}\right\}.$$

Then if $n \ge N$, $\left|\frac{3^n}{n!} - 0\right| = \frac{3^n}{n!} \le \frac{27}{2n} < \frac{27}{2N} < \varepsilon$

$$\Rightarrow \frac{3^n}{n!} \to 0 \quad \square$$

2a $x_n = \frac{n^2 - 3n + 7}{n^3 - 4n - 1}$ If we show that $x_{n+3} \to 0$ then it follows that $x_n \to 0$ also.

x_{n+3} (after simplification) $= \frac{n^2 + 3n + 7}{n^3 + 9n^2 + 23n + 14}$

hence $0 < x_{n+3} < \frac{n^2 + 3n + 7}{n^3} < \frac{n^2 + 3n^2 + 7n^2}{n^3} = \frac{11}{n}$

or $0 < x_{n+3} < 11/n$. $\therefore$ by Squeeze lemma, since $11/n \to 0$ and $\{0\} \to 0$

$$\Rightarrow x_{n+3} \to 0$$

$$\Rightarrow x_n \to 0$$

2b $x_n = \frac{n + \sin n}{2n}$

Since $-1 \le \sin n \le 1$ it follows that

$$\frac{n-1}{2n} \le x_n \le \frac{n+1}{2n}.$$

Since $\frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n} \to 1/2$

and $\frac{n-1}{2n} = \frac{1}{2} - \frac{1}{2n} \to 1/2$

note: these limits follow from "continuity of operations"

by Squeeze lemma, $x_n \to 1/2$

2.2.8 $\lim_{n\to\infty} \frac{n^2}{2^n} = 0$

ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}\right| = \left(\frac{n+1}{n}\right)^2 \cdot \frac{2^n}{2^{n+1}}$

$= \left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{2}$

$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right) \cdot \frac{1}{2} \to \frac{1}{2}$

($\frac{2}{n} \to 0$, $\frac{1}{n^2} \to 0$)

since $1/2 < 1 \Rightarrow a_n \to 0$

2.2.12 a) $\{a_n\}$ bdd, $b_n \to 0$ show $a_n b_n \to 0$

Since $\{a_n\}$ bdd, $\exists M \in \mathbb{R} : |a_n| \le M \quad \forall n$

Since $b_n \to 0$, $\forall \varepsilon > 0 \; \exists M_1 :$ if $n \ge M_1$

then $|b_n - 0| = |b_n| < \frac{\varepsilon}{M+1}$

Let $\varepsilon > 0$. Then if $n \ge M_1$

$|a_n b_n - 0| \le |a_n b_n| \le |a_n||b_n|$

$\le M|b_n|$

$< \frac{M\varepsilon}{M+1} < \varepsilon$

$\square$

b) $a_n = n^2 \quad b_n = 1/n$

$a_n b_n = n$ not convergent

c) $a_n = (-1)^n \quad b_n = 1$ all n

then $a_n b_n = (-1)^n$ not convergent