Comments on HW I

1.1.4 $S$ ordered set, $B \subset S$ bdd, $A \subset B$ nonempty;

$\inf A$, $\inf B$, $\sup B$, $\sup A$ exist show

$$\inf B \le \inf A \le \sup A \le \sup B$$
$$(1) \qquad (2) \qquad (3)$$

pf: Since $A \subset B$, any lower bound for $B$ is also a lower bound for $A$ $\Rightarrow$ $\inf B$ is a LB for $A$

$\Rightarrow GLB(A) = \inf A \ge \inf B \Rightarrow$ (1) holds

Let $a \in A$. Then any LB for $A$ is $\le a \Rightarrow \inf A \le a$

Likewise $a \le \sup A$. $\Rightarrow$ (2) holds.

The proof of (3) is similar to the proof of (1).

1.1.12 Any ordered field must contain a countably infinite set.

pf: We have proved $0<1$. Also by property (i) of ordered fields ($x,y,z \in \mathbb{F}$, $x<y \Rightarrow x+z < y+z$)

we have $0<1 \Rightarrow 0+1 < 1+1 \Rightarrow 1 < 1+1$.

repeating $1+1 < 1+1+1$, etc

$\Rightarrow 0 < 1 < 1+1 < 1+1+1 < \cdots$

this process creates a distinct field element at each step

$\therefore$ is in 1-1 correspondence with $\mathbb{N} \Rightarrow$ countable.

1.1.8. Let $F = \{0,1,2\}$ Prove there is exactly one way to define the operations. Since $0+x = x$ for $x \in F$ and $1 \cdot x = x$ for $x \in F$, and we proved that $0 \cdot x = 0$ for $x \in F$, the table for addition and multiplication is of the form

| + | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | ? | ? |
| 2 | 2 | ? | ? |

| · | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | ? |

Since 2 must have a multiplicative inverse $x$ so $2 \cdot x = 1$ and $x = 0, 1$ do not work $\Rightarrow x = 2$ is the only possibility.

$\Rightarrow 2 \cdot 2 = 1$

Since I allowed you to use $1+1=2$ (this can be proved also) this leaves $1+2$, $2+1$, and $2+2$ to determine.

By commutivity if $1+2 = a \Rightarrow 2+1 = a$. Let $2+2 = b$

If $2+1 = 1$ then $2+1+(-1) = 1+(-1) \Rightarrow 2 = 0 \rightarrow\leftarrow$

If $2+1 = 2$ then $2+1+(-2) = 2+(-2) \Rightarrow 1 = 0 \rightarrow\leftarrow$

$\Rightarrow 2+1 = 0 \quad \therefore a = 0$. Finally, $b = 2+2 = 2\cdot1 + 2\cdot1 = 2(1+1)$
$= 2\cdot2 = 1$

1.2.9 $A, B$ nonempty, bdd. $C = A+B$. Show $C$ bdd,

$\sup C = \sup A + \sup B$

pf $A, B$ bdd $\Rightarrow$ there are numbers $L_A, L_B, U_A, U_B$ so that for $a \in A$, ~~$L_A \le a$~~ $L_A \le a \le U_A$, and for $b \in B$ $L_B \le b \le U_B$

$\Rightarrow$ if $a \in A, b \in B \Rightarrow L_A + L_B \le a+b \le U_A + U_B$

$\therefore A+B$ is bdd.

Since for $a \in A, b \in B$ $a+b \le U_A + U_B$ where $U_A, U_B$ are upperbounds for $A, B \Rightarrow a+b \le \sup A + \sup B$

$\Rightarrow \sup A + \sup B$ is an UB for $A+B$

$\Rightarrow LUB(A+B) \le \sup A + \sup B$

or $\sup(A+B) \le \sup A + \sup B$

For reverse ineq., Let $\varepsilon > 0$. $\exists\, a \in A : \sup A - \frac{\varepsilon}{2} < a \le \sup A$

and $\exists\, b \in B : \sup B - \varepsilon/2 < b \le \sup B$

$\Rightarrow \exists\, a+b \in C : \sup A + \sup B - \varepsilon < a+b \le \sup A + \sup B$

$\therefore \sup(A+B) \ge a+b > \sup A + \sup B - \varepsilon$

$\Rightarrow \sup A + \sup B - \sup(A+B) \le \varepsilon \quad \forall \varepsilon > 0$

$\le 0$

$\Rightarrow$

$\Rightarrow \sup A + \sup B \le \sup(A+B)$ $\square$

1.2.15a Let $y \in \mathbb{R}$. Show $\sup \underbrace{\{x \in \mathbb{Q} : x < y\}}_{S} = y$

pf: $y$ is an UB of $S$ since if $x \in S \Rightarrow x < y. \Rightarrow \sup S \le y.$

If $\sup S < y$, then by density of $\mathbb{Q}$ $\exists$ $p/q \in \mathbb{Q} : \sup S < p/q < y$, but then $p/q \in S$ so $\sup S < p/q$ has been contradicted

$\Rightarrow \sup S = y.$ $\square$