Exact Controllability of an Elastic Membrane coupled with a Potential Fluid

Scott Hansen*

May 15, 2001

Abstract

We consider the problem of boundary control of an elastic system with coupling to a potential equation. The potential equation represents the linearized motions of an incompressible inviscid fluid in a cavity bounded in part by an elastic membrane. Sufficient control is placed on a portion of the elastic membrane to insure that the uncoupled membrane is exactly controllable. The main result is that if the density of the fluid is sufficiently small then the coupled system is exactly controllable.

Keywords: exact controllability, fluid-elastic interaction, fluid-structure interaction, potential fluid

1 INTRODUCTION

In this article we consider the problem of controlling an elastic membrane that is adjacent to a linear potential fluid. In this direction, there has recently been much research concerning the case of an elastic system with acoustic coupling (see e.g., (Avalos, 1996), (Banks et. al. 1997), (Lions and Zuazua, 1995), (Micu and Zuazua, 1997) and references therein). Other papers consider the case where the adjacent fluid is a Stokes fluid: (Osses, 1998), (Osses and Puel, 1999). In this paper the fluid is modeled as a linearized potential fluid (with a harmonic velocity potential). This model has been used for example, to analyze the dynamics of the cochlea in the inner ear (Lighthill, 1981). Some comparison of these various models can be found in (Conca et al.,1998).

The system we consider involves a body of fluid bounded at least partly by a flexible membrane. A potential equation is used to model the fluid while a wave equation is used to model the membrane. The two equations are coupled by matching velocities of the fluid with that of the membrane, and using the fluid pressure as

*Department of Mathematics, Iowa State University, Ames, Iowa 50011 USA, email:shansen@iastate.edu

a forcing term for the membrane. Furthermore, the incompressibility of the fluid introduces constraints upon the possible motions of the membrane.

It is well known for the wave equation that Dirichlet boundary control on a "sufficiently large" portion of the boundary is sufficient for exact controlliblity. The main result of this article is that if the fluid density is sufficiently small, the same holds true for the coupled fluid-membrane system. A similar result, but for some special cases involving a two-dimensional potential fluid *surrounding* a one-dimensional elastic system appeared in (Hansen and Lyashenko, 1997). There, the moment method was applied and hence that approach does not work for the case of a three-dimensional fluid considered here. Here, a modification of the classical "multiplier method" is applied. The modifications involve handling the additional terms from the fluid coupling, and also finding suitable multipliers that are valid in consideration of the incompressibility constraint.

1.1 Problem Formulation.

Consider the situation of a fluid in a cavity in which a portion (at least) of the boundary is flexible. Thus the domain of the fluid has a boundary consisting of a rigid part and a flexible part such that the fluid is on one side of the flexible boundary. (This requirement is for simplicity only.) The fluid in the cavity is assumed to be incompressible, irrotational (inviscid), and velocities are small enough so that linearization about the motionless state is valid. The membrane is forced by the pressure of the fluid and the velocity of the fluid is matched with the velocity of the boundary. Control is exercised on a portion of the boundary of the membrane.

To describe the situation mathematically we let Ω denote a bounded domain in $\mathbf{R}^3$ ($\mathbf{R}^2$ is OK, with obvious adjustments) with Lipschitz boundary Γ. It is assumed that Γ consists of an *inflexible* part Γ_0 and and a *flexible* part ω. For simplicity it is assumed that at equilibrium, ω is a subdomain of the plane $x_3 = 0$ that has a sufficiently smooth (C^2 is fine) boundary $\gamma \neq \emptyset$, which itself consists of a controlled part γ_1 and an uncontrolled part γ_0. To avoid a discussion of singularities, we assume that $\gamma = \gamma_0 \cup \gamma_1$ with $\bar{\gamma}_0 \cap \bar{\gamma}_1 = \emptyset$. We also need some condition that guarantees that control is active on a large enough portion of γ. One can assume γ_0 to be empty, otherwise it is assumed that the uncontrolled region satisfies the standard geometric condition:

$$(x_1, x_2) \cdot \mathbf{n} \leq 0 \text{ on } \gamma_0 \tag{1.1}$$

where $\mathbf{n}$ denotes the unit outward normal vector to γ (in the plane $x_3 = 0$).

Due to the incompressibility, the fluid velocity $\mathbf{q}$ satisfies

$$\operatorname{div} \mathbf{q} = 0, \text{ on } \Omega \times \mathbf{R}^+$$

and the assumption that the fluid is irrotational (inviscid) implies that

$$\operatorname{curl} \mathbf{q} = 0 \text{ on } \Omega \times \mathbf{R}^+.$$

Consequently

$$\mathbf{q} = \nabla \Phi, \qquad \text{where } \Delta \Phi = 0 \text{ on } \Omega \times \mathbf{R}^+. \tag{1.2}$$

Matching velocities of the fluid and membrane at $x_3 = 0$ leads to

$$\frac{\partial \Phi}{\partial \mathbf{n}} = \begin{cases} 0 & \text{on } \Gamma_0 \times \mathbf{R}^+ \\ w_t & \text{on } \omega \times \mathbf{R}^+. \end{cases} \tag{1.3}$$

Actually, if velocities are matched on the deformed boundary ω we obtain a free boundary problem. However since small vibrations are under consideration we linearize by matching velocities on the boundary of the (fixed) equilibrium domain Ω. Henceforth, Ω, and its boundary will be assumed to be fixed, as is the case in all the papers mentioned earlier.

The energy $\mathcal{E}(t)$ is the sum of the kinetic $\mathcal{K}(t)$ and potential $\mathcal{P}(t)$ energy where

$$\mathcal{K} = \frac{1}{2}\int_\Omega \rho|\nabla_3\Phi|^2\, d\Omega + \frac{1}{2}\int_\omega |w_t|^2\, d\omega, \qquad \mathcal{P} = \frac{1}{2}\int_\omega |\nabla_2 w|^2\, d\omega,$$

where ∇_k denotes the gradient in the first k coordinate directions.

The equations of motion can be obtained from Hamilton's principle. That is, the first variation, with respect to a class of admissible variations, of the Lagrangian $\mathcal{L} = \int_0^T \mathcal{K} - \mathcal{P}\, dt$ is set to zero. The class of variation functions $\{\hat{w}, \hat{\Phi}\}$ we consider are those that satisfy the constraints (1.2), (1.3), $\hat{w}|_\gamma = 0$, and vanish near $t = 0$ and $t = T$. We obtain

$$\begin{aligned} 0 &= \textstyle\int_0^T \{\int_\omega w_t\hat{w}_t - \nabla w\nabla\hat{w}\, d\omega + \rho\int_\Omega \nabla\Phi\nabla\hat{\Phi}\, d\Omega\}\, dt \\ &= \textstyle\int_0^T \int_\omega \{(-w_{tt} + \Delta w)\hat{w} + \Phi\hat{w}_t\}\, d\omega\, dt \\ &= \textstyle\int_0^T \int_\omega \{(-w_{tt} + \Delta w - \rho\Phi_t)\hat{w}\, d\omega\}\, dt. \end{aligned} \tag{1.4}$$

Note that since $\hat{\Phi}$ is determined by a Neumann problem, $\int_\omega \hat{w}_t\, d\omega = \int_\omega \partial\hat{\Phi}/\partial x_3 = 0$. Therefore $\int_\omega \hat{w}\, d\omega$ is constant and equal to its initial value. Therefore

$$\int_\omega \hat{w}\, d\omega = 0 \qquad \forall t \geq 0.$$

Consequently the equations (in strong form) are only determined up to an additive constant (denoted by C).

In strong form the equations of motion become

$$w_{tt} + \rho\Phi_t - \Delta w = C \qquad \text{in } \omega \times \mathbf{R}^+ \tag{1.5}$$

$$\Delta\Phi = 0 \qquad \text{in } \Omega \times \mathbf{R}^+ \tag{1.6}$$

$$\frac{\partial\Phi}{\partial\mathbf{n}} = \begin{cases} 0 & \text{on } \Gamma_0 \times \mathbf{R}^+ \\ w_t & \text{on } \omega \times \mathbf{R}^+ \end{cases} \tag{1.7}$$

$$w = \begin{cases} 0 & \text{on } \gamma_0 \times \mathbf{R}^+ \\ f & \text{on } \gamma_1 \times \mathbf{R}^+ \end{cases} \tag{1.8}$$

Initial conditions are of the form

$$(w, w_t)|_{t=0} = (w_0, w_1) \quad \text{where } \int_\omega w_1\, d\omega = 0. \tag{1.9}$$

Due to (1.7) we have that $\int_\omega w\, d\omega = \int_\omega w_0\, d\omega$. One can rewrite the equations in terms of $\tilde{w} = w - w^*$, where w^* is the steady state solution determined by (1.5) and (1.8) (with zero boundary data) and the condition $\int_\omega w^*\, d\omega = \int_\omega w_0$. Thus, without loss of generality,

$$\int_\omega w_0\, d\omega = 0. \tag{1.10}$$

The natural energy space E for the system is

$$E = \{(w, w_t, \Phi) \in \tilde{H}_0^1(\omega) \times \tilde{L}^2(\omega) \times (H^1(\Omega)/C) : \Phi \text{ satisfies (1.6), (1.7)}\} \qquad (1.11)$$

where $\tilde{H}_0^1(\omega)$ and $\tilde{L}^2(\omega)$ denote the subspaces of functions in $H_0^1(\omega)$ and $L^2(\omega)$ (respectively) that are orthogonal to constants. The space $H^1(\Omega)/C$ denotes the equivalence classes of functions in $H^1(\Omega)$ that are identified up to an additive constant.

Concerning the uncontrolled system we will prove the following.

Theorem 1.1 *Assume that $f = 0$ and $(w_0, w_1) \in \tilde{H}_0^1(\omega) \times \tilde{L}^2(\omega)$. Then (1.5)–(1.8), (1.9) has a unique solution with*

$$(w, w_t, \Phi) \in C([0, \infty); E). \qquad (1.12)$$

Moreover the energy $\mathcal{E}(t) = \mathcal{K}(t) + \mathcal{P}(t)$ is conserved along solution trajectories. If in addition $(w_0, w_1) \in \mathcal{V} := H^2(\omega) \cap \tilde{H}^1(\omega) \times \tilde{H}_0^1(\omega)$ then

$$(w, w_t) \in C([0, \infty); \mathcal{V}). \qquad (1.13)$$

We will however wish to utilize a control $f \in L^2((0,T) \times \gamma_1)$ and hence it is necessary to work with weak solutions defined on $\mathcal{V}'$ (the dual of $\mathcal{V}$ relative to an inner product defined on $\mathcal{H}$; see Section 2). Later we show that for $f \in L^2((0,T) \times \gamma_1)$ and any initial condition $(w_0, w_1) \in \mathcal{V}'$ there is a unique weak solution that satisfies

$$(w, w_t) \in C([0, T); \mathcal{V}').$$

Regarding our control problem, under all the geometric conditions described earlier, we have the following main result.

Theorem 1.2 *There exists $\rho_0 > 0$ such that if $0 \leq \rho < \rho_0$ then the system (1.5)–(1.8) is exactly controllable on the space $\tilde{L}^2(\omega) \times \tilde{H}^{-1}(\omega)$. That is, for T large enough, if $0 \leq \rho < \rho_0$, given any initial data $\{w_0, w_1\} \in \mathcal{V}'$ there exists an $f \in L^2((0,T) \times \omega)$ such that $\{w, w_t\}|_{t=T} = \{0, 0\}$ and $\Phi|_{t=T}$ is constant.*

Concerning Theorem 1.2 we have the following remarks.

Remark 1.1 Of course, due to the time-reversibility of the system, equivalently one can find an L^2 control drives the given initial state to any desired terminal state in $\mathcal{V}'$ in the same time.

Remark 1.2 Actually we will prove the following "observability inequality" which by duality (i.e., "Hilbert's Uniqueness Method", (Lions, 1998)) is equivalent to the controllability in Theorem 1.2: *Let $\{w^0, w^0\} \in \tilde{H}_0^1(\omega) \times \tilde{L}^2(\omega)$ and suppose T and ρ are as in Theorem 1.2. Let w denote a solution to (1.5)–(1.10) with $f = 0$. Then there exists $c > 0$ such that*

$$\int_0^T \int_{\gamma_1} |\frac{\partial w}{\partial \mathbf{n}}|^2 \, d\gamma \, dt \geq c\mathcal{E}(0). \qquad (1.14)$$

Remark 1.3 It unknown whether ρ_0 in Theorem 1.2 can be taken arbitrarily large, or whether the control time T is as small as that of the uncoupled wave equation on ω. On the other hand, the proof provides an explicit lower bound for ρ_0 in terms of the geometry of Ω and an explicit estimate for T in terms of ρ and the geometry of Ω.

2 Existence, Uniqueness, Regularity

In this section we prove the well-posedness of the coupled elastic-fluid system.

2.1 Regularity of fluid pressure on beam.

We first discuss some properties of the Neumann problem for the Laplacian. We assume that Ω is a bounded domain in $\mathbf{R}^3$ with boundary Γ.

Consider the following Neumann problem.

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega \\ \frac{\partial\phi}{\partial\mathbf{n}} = f & \text{on } \Gamma \end{cases} \tag{2.1}$$

The solvability condition for (2.1) is

$$\int_\Gamma f\, d\Gamma = 0. \tag{2.2}$$

It is well known that when Γ and f are regular, and f satisfies (2.2), there exist classical solutions to (2.1) which are unique up to an arbitrary additive constant. If f or Γ is less regular then variational solutions may be defined in some cases. For our purposes the following regularity results (Necas, 1967) will be sufficient.

Proposition 2.1 *Concerning (2.1), (2.2), suppose Γ is Lipschitz and $-1 \le s \le 0$. If $f \in H^s(\Gamma)$, (with zero average) then $\phi \in H^{s+3/2}(\Omega)/C$ and $\phi|_\Gamma \in H^{s+1}(\Gamma)/C$, where X/C denotes the quotient space of functions in X identified up to an additive constant.*

For $\phi \in \tilde{L}^2(\omega)$ define the *Neumann to Dirichlet map* Λ by

$$\Lambda\phi = \Phi|_\omega,$$

where

$$\begin{aligned} \Delta\Phi &= 0 && \text{in } \Omega \\ \Phi &= 0 && \text{on } \Gamma_0 \\ \Phi &= \phi && \text{on } \omega. \end{aligned}$$

We can prove the following.

Proposition 2.2 *Assume that Γ is Lipschitz. Then $\Lambda : \tilde{L}^2(\omega) \to H^1(\omega)/C$ continuously and satisfies*

$$\|\Lambda\phi\|_{\tilde{H}^1(\omega)} \le C_\Omega \|\phi\|_{L^2(\omega)} \qquad \forall\, \phi \in \tilde{L}^2(\omega); \tag{2.3}$$

Furthermore, Λ is positive and self-adjoint on $\tilde{L}^2(\omega)$ in the sense that

$$\int_\omega (\Lambda\phi)\psi\, d\omega = \int_\omega (\Lambda\psi)\phi\, d\omega \qquad \forall\, \phi, \psi \in \tilde{L}^2(\omega); \tag{2.4}$$

$$\int_\omega (\Lambda\phi)\phi\, d\omega \ge 0 \qquad \forall\, \phi \in \tilde{L}^2(\omega). \tag{2.5}$$

Proof: The estimate (2.3) follows immediately from Proposition 2.1. To prove the second part, let let F and G be harmonic functions on Ω with

$$\frac{\partial F}{\partial \mathbf{n}} = \begin{cases} 0 & \text{on } \Gamma_0 \\ \phi & \text{on } \omega \end{cases} \quad ; \qquad \frac{\partial G}{\partial \mathbf{n}} = \begin{cases} 0 & \text{on } \Gamma_0 \\ \psi & \text{on } \omega. \end{cases} \tag{2.6}$$

An integration by parts, using the definition of Λ shows

$$\int_\omega (\Lambda\phi)\psi \, d\omega = \int_\omega F \frac{\partial G}{\partial \mathbf{n}} \, d\omega = \int_\Omega \nabla F \nabla G \, d\omega = \int_\omega \phi \Lambda \psi \, d\omega. \tag{2.7}$$

The same equations with $\psi = \phi$ establishes the positivity in (2.5). This completes the proof. □

We can now, at least formally, rewrite the system (1.5)–(1.8) as

$$w_{tt} + \rho(\Lambda w_t)_t - \Delta w = C \qquad \text{in } \omega \times \mathbf{R}^+ \tag{2.8}$$

$$w = \begin{cases} 0 & \text{on } \gamma_0 \times \mathbf{R}^+ \\ f & \text{on } \gamma_1 \times \mathbf{R}^+. \end{cases} \tag{2.9}$$

2.2 Finite energy solutions.

In this section we use the semigroup theory to prove the existence and uniqueness of finite energy solutions to (2.8)–(2.9).

Let us define $\mathcal{C}_\rho : \tilde{L}^2(\omega) \to L^2(\omega)/C$ by

$$\mathcal{C}_\rho \phi = \phi + \rho \Lambda \phi.$$

From the previous proposition it is clear that $\mathcal{C}_\rho$ is positive and self-adjoint for $\rho \geq 0$ in the same sense Λ is in (2.4), (2.5). For this reason it is convenient to identify the space $\tilde{L}^2(\omega)$ with its dual $L^2(\omega)/C$. Under this identification each element $\phi \in L^2(\omega)/C$ is identified with an element $\phi_0 \in \tilde{L}^2(\omega)$ by $\phi_0 = \phi + C_0$, where C_0 is picked so that $\int_\omega \phi + C_0 \, d\omega = 0$. This way $\mathcal{C}_\rho$ is positive and self-adjoint on $\tilde{L}^2(\omega)$ in the usual sense.

Define the following forms

$$\begin{aligned} (\phi, \psi) &= \int_\omega \phi \cdot \bar{\psi} \, d\omega \qquad \forall \phi, \psi \in (L^2(\omega))^3, \\ < \phi, \psi >_\rho &= \int_\omega (\mathcal{C}_\rho \phi) \bar{\psi} \, d\omega \qquad \forall \phi, \psi \in \tilde{L}^2(\omega) \\ < \{\phi_1, \phi_2\}, \{\psi_1, \psi_2\} >_\mathcal{E} &= (\nabla \phi_1, \nabla \psi_1) + < \phi_2, \psi_2 >_\rho \qquad \forall \{\phi_1, \phi_2\}, \{\psi_1, \psi_2\} \in \mathcal{H}. \end{aligned}$$

Due to Poincaré's inequality and the above described properties of the operator $\mathcal{C}_\rho$, $(\nabla \cdot \, , \nabla \cdot)$ is easily seen to be an inner product on $\tilde{H}^1_0(\omega)$. Likewise $< \cdot \, , \cdot >_\rho$ is an inner product on $\tilde{L}^2(\omega)$. Consequently $< \cdot \, , \cdot >_\mathcal{E}$ is an inner product on the finite energy space $\mathcal{H}$.

Define $y = w$ and $v = w_t$. The first order form of (2.8) is

$$\frac{d}{dt} \begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & I \\ \mathcal{C}_\rho^{-1} \Delta_C & 0 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} =: \mathcal{A}\{y, v\} \tag{2.10}$$

where we have written Δ_C to emphasize that the range must be considered as a quotient space - or equivalently, we put $\Delta_C \phi = \Delta\phi + C$ such that $\Delta\phi + C$ has zero average (as is done when $\tilde{L}^2(\omega)$ is identified with its dual $L^2(\omega)/C$).

Also define the spaces

$$\mathcal{H} = \tilde{H}^1_0(\omega) \times \tilde{L}^2(\omega), \qquad \mathcal{V} = H^2(\omega) \cap \tilde{H}^1_0(\omega) \times \tilde{L}^2(\omega).$$

Proposition 2.3 *Let $\rho > 0$. The operator $\mathcal{A}$ is the generator of a strongly continuous group of isometries with respect to the energy inner product $< \cdot, \cdot >_{\mathcal{E}}$ on the finite energy space $\mathcal{H}$. Consequently given the initial conditions*

$$\{y(0), v(0)\} = \{y^0, v^0\} \in \mathcal{H} \tag{2.11}$$

there exists a unique solution $\{y, v\}$ to (2.10) that satisfies

$$\{y, v\} \in C(\mathbf{R}; \mathcal{H}). \tag{2.12}$$

Proof: With $\mathcal{D}(\mathcal{A}) = \mathcal{V}$ it is clear that $\mathcal{A}$ is densely defined. Let us verify that $\mathcal{A}$ is closed. We let $\{\phi_n, \psi_n\} \to \{\phi, \psi\}$ in $\mathcal{H}$, with each $\{\phi_n, \psi_n\} \in \mathcal{V}$. Assume that $\{y_n, v_n\} := \mathcal{A}\{\phi_n, \psi_n\} \to \{y, v\}$ in $\mathcal{H}$. We need to see that $\{\phi, \psi\} \in \mathcal{V}$ and $\mathcal{A}\{\phi, \psi\} = \{y, v\}$. Since $y_n = \psi_n \to y$ in $\tilde{H}^1_0(\omega)$ and $\psi_n \to \psi$ in $\tilde{L}^2(\omega)$ it follows that $\psi = y \in \tilde{H}^1_0(\omega)$. For the other variable we have that $v_n = \mathcal{C}_\rho^{-1}\Delta_C\phi_n \to v$ in $\tilde{L}^2(\omega)$ and $\phi_n \to \phi$ in $\tilde{H}^1_0(\omega)$. First note that $-\Delta_C$ is associated with a coercive symmetric quadratic form on $\tilde{H}^1_0(\omega)$ since for all f, g in $H^2(\omega) \cap \tilde{H}^1_0(\omega)$ one has

$$\int_\omega -\Delta_C f\, g\, d\omega = \int_\omega \nabla f \cdot \nabla g\, d\omega = -\int_\omega f \Delta_C g\, d\omega.$$

The form is strictly positive since

$$\min_{u \in \tilde{H}^1_0(\omega)} \frac{(\nabla u, \nabla u)}{\|u\|^2_{L^2}} \geq \min_{u \in H^1_0(\omega)} \frac{(\nabla u, \nabla u)}{\|u\|^2_{L^2}} = \lambda_1 > 0$$

where λ_1 is the first eigenvalue of $-\Delta$ (the usual Laplacian operator with Dirichlet boundary conditions). It follows from the Lax-Milgram theorem that Δ_C is an isomorphism from $H^2(\omega) \cap \tilde{H}^1_0(\omega)$ to $\tilde{L}^2(\omega)$. Due to positivity of Λ, $\mathcal{C}_\rho$ is also an isomorphism on $\tilde{L}^2(\omega)$. From this we conclude that $\{\phi_n\}$ is convergent in $H^2(\omega)$. Since $\phi_n \to \phi$ in $\tilde{H}^1_0(\omega)$ we see that $\phi = \lim_{n\to\infty} \Delta_C^{-1}\mathcal{C}_\rho v_n = \Delta_C^{-1}\mathcal{C}_\rho v$. Thus $\mathcal{A}$ is closed.

To show that $\mathcal{A}$ generates a group we apply the Lumer-Phillips theorem to $\mathcal{A}$ and $-\mathcal{A}$, i.e., it is enough to show that $\mathcal{A}$, $\mathcal{A}^*$, $-\mathcal{A}$, $-\mathcal{A}^*$ are all dissipative. However, this follows if we show that $\mathcal{A}$ is anti-Hermitian relative to the energy inner product. We calculate for all $\{y_i, v_i\} \in \mathcal{V}$, $i = 1, 2$:

$$\begin{aligned} < \mathcal{A}\{y_1, v_1\}, \{y_2, v_2\} >_{\mathcal{E}} &= < \{v_1, \mathcal{C}_\rho^{-1}\Delta_C y_1\}, \{y_2, v_2\} >_{\mathcal{E}} & (2.13)\\ &= (\nabla v_1, \nabla y_2) + < \mathcal{C}_\rho^{-1}\Delta_C y_1, v_2 >_\rho \\ &= -(v_1, \Delta y_2) + (\Delta_C y_1, v_2) = -(v_1, \Delta_C y_2) + (\Delta y_1, v_2) \\ &= -(\mathcal{C}_\rho v_1, \mathcal{C}_\rho^{-1} y_2) - (\nabla y_1, \nabla v_2) \\ &= - < v_1, \mathcal{C}_\rho^{-1} y_2 >_\rho - (\nabla y_1, \nabla v_2) \\ &= - < \{y_1, v_1\}, \mathcal{A}\{y_2, v_2\} >_{\mathcal{E}} & (2.14) \end{aligned}$$

Thus $\mathcal{A}$ is antisymmetric and $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A}^*)$. However, as we have previously shown, $\mathcal{C}_\rho \Delta_C$ is an isomorphism from $H^2(\omega) \cap \tilde{H}_0^1(\omega)$ to $\tilde{L}^2(\omega)$, it follows that $\mathcal{A}$ is surjective from $\mathcal{V}$ to $\mathcal{H}$ and hence $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}^*)$. Thus the semigroup generated by $\mathcal{A}$ is actually a group, and is easily shown to be unitary. The other statements in the theorem are immediate consequences. □

2.3 Weak solutions.

Let us denote the dual space to $\tilde{H}_0^1(\omega)$ relative to $< \cdot, \cdot >_\rho$ by $\tilde{H}^{-1}(\omega)$. Since we have proven that $\mathcal{A} : \mathcal{V} \to \mathcal{H}$ is one to one and onto, when $\mathcal{V}$ is endowed with the graph norm: $\|\{y,v\}\|_{\mathcal{V}} = \|\mathcal{A}\{y,v\}\|_{\mathcal{E}}$, $\mathcal{A}$ becomes a topological isomorphism as well. We can define by duality an extension of $\mathcal{A}$, temporarily denoted $\hat{\mathcal{A}}$, from $\mathcal{H}$ to $(\mathcal{D}(\mathcal{A}^*))' = (\mathcal{D}(\mathcal{A}))' =: \mathcal{V}'$ by

$$< \hat{\mathcal{A}}\{y,v\}, \{Y,V\} >_{\mathcal{E}} = < \{h,v\}, -\mathcal{A}\{Y,V\} >_{\mathcal{E}} \qquad \forall \{Y,V\} \in \mathcal{V}.$$

Since $(\nabla y, \nabla Y) = (y, \Delta_C Y)$ for all $y \in \tilde{H}_0^1(\omega)$ and all $Y \in H^2(\omega) \cap H_0^1(\omega)$ we see that the first component of $(\mathcal{D}(\mathcal{A}))'$ is $\tilde{L}^2(\omega)$. The second component is the dual space to $\tilde{H}_0^1(\omega)$ relative to $< \cdot, \cdot >_\rho$. Denote this by $\tilde{H}^{-1}(\omega)$.

The extended operator $\hat{\mathcal{A}}$ can be shown to be the generator of a strongly continuous semigroup of unitary operators isomorphic to the original one. Henceforth we make no distinction between $\mathcal{A}$ and its possible extensions. Concerning the system (2.10), or equivalently (2.8)-(2.9) with $f = 0$ we have the following.

Corollary 2.1 *The semigroup defined in Proposition 2.3 extends continuously to a strongly continuous, unitary group on the space* $\mathcal{V}' := \tilde{L}^2(\omega) \times \tilde{H}^{-1}(\omega)'$. *Consequently, given the initial data* $\{y^0, v^0\} \in \mathcal{V}'$ *there is a uniquely defined solution to (2.8)-(2.9) with* $f = 0$ *which satisfies*

$$y \in C((-\infty,\infty), \tilde{L}^2(\omega)) \cap C^1((-\infty,\infty), \tilde{H}^{-1}(\omega)').$$

The nonhomogeneous system we are interested in is

$$\mathcal{C}_\rho w_{tt} - \Delta w = C \quad \text{in } \omega \times (0,\infty) \tag{2.15}$$
$$w = f \quad \text{on } \gamma_0 \times (0,\infty) \tag{2.16}$$
$$w = 0 \quad \text{on } \gamma_1 \times (0,\infty) \tag{2.17}$$
$$\{w, w_t\}|_{t=0} = \{w^0, w^1\} \in \tilde{L}^2(\omega) \times \tilde{H}^{-1}(\omega). \tag{2.18}$$

To define a weak solution we write $w = w_0 + z$ where w_0 satisfies (2.15)–(2.18) with $f = 0$ in (2.16) and z satisfies (2.15)–(2.18) but with $\{w^0, w^1\} = \{0, 0\}$. Formally multiplying (2.15) (with z in place of w) by a smooth function ϕ with

$$\phi \in C([0,T], H^2(\omega) \cap \tilde{H}_0^1(\omega)) \cap C^1([0,T], \tilde{H}_0^1(\omega))$$

results in

$$\begin{aligned}
0 &= \int_0^T \int_\omega C\phi \, dx\, dt = \int_0^T \int_\omega \mathcal{C}_\rho z_{tt}\phi - \Delta z \phi \, dx\, dt \\
&= \int_\omega \mathcal{C}_\rho z_t(T)\phi(T) - \mathcal{C}_\rho z(T)\phi_t(T)\, dx\, dt + \int_0^T \int_{\gamma_0} f \frac{\partial \phi}{\partial \mathbf{n}}\, d\gamma\, dt \\
&\quad + \int_{Q_T} z\mathcal{C}_\rho \phi_{tt} - z\Delta\phi \, dx\, dt.
\end{aligned}$$

Hence if ϕ is a solution the *dual system*:

$$\mathcal{C}_\rho \phi_{tt} - \Delta\phi = C \quad \text{in } \omega \times (-\infty, \infty) \tag{2.19}$$

$$\phi = 0 \quad \text{on } \gamma_0 \times (-\infty, \infty) \tag{2.20}$$

$$\phi = 0 \quad \text{on } \gamma_1 \times (-\infty, \infty) \tag{2.21}$$

$$\{\phi, \phi_t\}|_{t=T} = \{\phi^0, \phi^1\} \tag{2.22}$$

with $\{\phi^0, \phi^1\} \in H^2(\omega) \cap \tilde{H}_0^1(\omega) \times \tilde{H}_0^1(\omega)$, we obtain the identity

$$< z_t(T), \phi^0 >_\rho - < z(T), \phi^1 >_\rho = -\int_0^T \int_{\gamma_0} f \frac{\partial \phi}{\partial \mathbf{n}}\, d\gamma\, dt. \tag{2.23}$$

We see that when $f \in L^2((0,T) \times (\gamma_0))$ we have

$$\begin{aligned}
\left| \int_0^T \int_{\gamma_0} f \frac{\partial \phi}{\partial \mathbf{n}}\, d\gamma\, dt \right| &\leq K \|f\|_{L^2(\gamma_0 \times (0,T))} \left\| \frac{\partial \phi}{\partial \mathbf{n}} \right\|_{L^2(\gamma_0 \times (0,T))} \\
&\leq K \|f\|_{L^2(\gamma_0 \times (0,T))} \|\{\phi^0, \phi^1\}\|_{\mathcal{V}}
\end{aligned}$$

where K is a constant that may change from line to line and we have used Proposition 2.3. Consequently the right hand side of (2.23) can be viewed as a continuous linear functional acting on the space $H^2(\omega) \cap \tilde{H}_0^1(\omega) \times \tilde{H}_0^1(\omega)$. Therefore by considering all possible $\{\phi^0, \phi^1\} \in H^2(\omega) \cap \tilde{H}_0^1(\omega) \times \tilde{H}_0^1(\omega)$ the identity (2.23) defines a unique element $\{z(T), z_t(T)\}$ as an element of the dual space. If we define $\tilde{H}^{-2}(\omega) = (H^2(\omega) \cap \tilde{H}_0^1(\omega))'$, where the duality is with respect to $< \cdot, \cdot >_\rho$ we therefore see, considering all $t : 0 \leq t \leq T$ that (2.23) defines a unique solution z with $z \in C([0,T], \tilde{H}^{-1}(\omega)$ and $z_t \in L^\infty(H^{-2}(\omega))$. Results on "regularity lifting" actually provide that $z_t \in C([0,T], H^{-1}(\omega))$. We therefore have the following (suboptimal) result, which will be improved shortly.

Proposition 2.4 *Let* $\{w^0, w^1\} \in \mathcal{V}'$ *and* $f \in L^2((0,T) \times \gamma_0)$. *Then there is a unique weak solution* w *to (2.15)–(2.18) for which*

$$w \in C([0,T], \tilde{H}^{-1}(\omega)) \cap C^1([0,T], \tilde{H}^{-2}(\omega)). \tag{2.24}$$

3 Optimal regularity

We will see that the regularity in (2.24) of Proposition 2.4 can actually be improved one degree. This occurs in the (uncoupled) wave equation with Dirichlet control and the usual argument involves multiplying the wave equation $w_{tt} - \Delta w = 0$ through by the multiplier $h \cdot \nabla w$, where h is a $C^2(\bar{\omega})$ vector field which is equal to the normal to γ on γ. However, in the present situation the equation of motion occurs in a quotient space and hence, in addition to the previous conditions on the multiplier $h \cdot \nabla w$, we need that $\int_\omega h \cdot \nabla w \, dx = 0$. For smooth solutions, $w = 0$ on γ and hence, equivalently we need that $\int_\omega w \nabla h \, dx = 0$ for each solution w. A sufficient condition, since solutions w have zero average value, is that $\int_\omega \nabla \cdot h \, dx = C$, where C is any constant. Fortunately, the problem of obtaining such a multiplier h is solved in (Galdi, 1994). From (Galdi, 1994) we have the following:

Lemma 3.1 *Under our assumptions on ω (ω is a bounded domain in $\mathbf{R}^2$ with C^2 boundary) there exists a vector field $h : \bar{\omega} \to \mathbf{R^2}$ such that*

(i) h is C^2 on $\bar{\omega}$

(ii) $h = \mathbf{n}$ on γ, where $\mathbf{n}$ is the outward unit normal vector to γ.

(iii) $\nabla \cdot h = C$ on ω, where C is a constant.

Using the multiplier from Lemma 3.1 applied to the homogeneous system (2.19)–(2.22) we obtain the following.

Proposition 3.1 *Let ϕ be a solution to the homogeneous backwards problem (2.19)–(2.22) with the data $\{\phi^0, \phi^1\}$ given in $\mathcal{V}$. Then there exists $K_1 > 0$, $K_2 > 0$ independent of T and the data, such that*

$$\int_0^T \int_\gamma |\frac{\partial \phi}{\partial \mathbf{n}}|^2 \leq (K_1 + K_2 T) \|\{\phi^0, \phi^1\}\|^2_{\mathcal{E}}. \tag{3.1}$$

Proof: Let us make the notation

$$\int\int \psi \;=\; \int_{\omega \times (0,T)} \psi \, d\omega \, dt, \qquad \int \psi \;=\; \int_{\gamma \times (0,T)} \psi \, d\gamma \, dt. \tag{3.2}$$

Also denote

$$X = (\phi_t + \rho \Lambda \phi_t, h \cdot \nabla \phi)|_0^T, \qquad Y = (\phi_t + \Lambda \rho \phi_t, \phi)|_0^T.$$

where $(\psi, \phi) \;=\; \int_\omega \psi \phi \, d\omega$

Let ϕ be as in the hypothesis and h as in Lemma 3.1. We calculate using Lemma 3.1 and integrations by parts the following

$$\begin{aligned} 0 \;&=\; \int\int \{(\mathcal{C}_\rho \phi_{tt} - \Delta \phi + C)(h \cdot \nabla \phi)\} \\ &=\; X - \int \{\frac{\partial \phi}{\partial \mathbf{n}} h \cdot \nabla \phi\} + \int\int \{\nabla \phi \nabla (h \cdot \nabla \phi) - \mathcal{C}_\rho \phi_t (h \cdot \nabla \phi_t)\} \\ &=\; X - \int \{|\frac{\partial \phi}{\partial \mathbf{n}}|^2\} + \int\int \{\nabla \phi \nabla (h \cdot \nabla \phi) - \mathcal{C}_\rho \phi_t (h \cdot \nabla \phi_t)\}. \end{aligned} \tag{3.3}$$

Let us define the gradient of a vector $h = (h_i)$ to be the matrix $\nabla h = ((\nabla h)_{ij})$ where $(\nabla h)_{ij} = \partial h_i / \partial x_j$. Then we have

$$
\begin{aligned}
\int\int \{\nabla\phi \cdot \nabla(h\cdot\nabla\phi)\} &= \int\int \{\nabla\phi\cdot(\nabla h \nabla\phi) + \frac{1}{2}\nabla|\nabla\phi|^2 \cdot h\} \\
&= \int\int \{\nabla\phi\cdot(\nabla h\nabla\phi) - \frac{1}{2}(\nabla\cdot h)|\nabla\phi|^2\} + \frac{1}{2}\int |\nabla\phi|^2 \\
&= \int\int \{\nabla\phi\cdot(\nabla h\nabla\phi) - \frac{1}{2}(\nabla\cdot h)|\nabla\phi|^2\} + \frac{1}{2}\int |\frac{\partial\phi}{\partial\mathbf{n}}|^2 \qquad (3.4)
\end{aligned}
$$

where we have used the fact that ϕ vanishes on γ in the last line.

Define the operator $\mathcal{M}$ initially on $\tilde{H}_0^1(\omega)$ by

$$
\mathcal{M}\psi = \Lambda(h\cdot\nabla\psi) - h\cdot\nabla\Lambda\psi. \qquad (3.5)
$$

We also have that

$$
\begin{aligned}
\int\int (\mathcal{C}_\rho\phi_t)h\cdot\nabla\phi_t &= \int\int(\phi_t + \rho\Lambda\phi_t)h\cdot\nabla\phi_t \\
&= \int\int\{\frac{1}{2}h\cdot\nabla\phi_t^2 + \frac{\rho}{2}h\cdot\nabla(\phi_t\Lambda\phi_t) \\
&\quad +\frac{\rho}{2}(h\cdot\nabla\phi_t)\Lambda\phi_t - \phi_t h\cdot\nabla(\Lambda\phi_t)\} \\
&= \frac{-C_1}{2}\left[\int\int\{\phi_t^2 + \rho(\Lambda\phi_t)\phi_t\}\right] + \frac{\rho}{2}\int\int \phi_t\mathcal{M}\phi_t \\
&= \int_0^T \frac{-C_1}{2} < \phi_t, \phi_t >_\rho + \frac{\rho}{2}(\phi_t, \mathcal{M}\phi_t)\, dt. \qquad (3.6)
\end{aligned}
$$

In the previous lines, C_1 is the constant of Lemma 3.1. Putting (3.4) and (3.6) together we obtain

$$
\begin{aligned}
\int |\frac{\partial\phi}{\partial\mathbf{n}}|^2 &= 2X + 2\int\int\{\nabla\phi\cdot(\nabla h)\nabla\phi\} \\
&\quad + C_1\int_0^T\{< \phi_t, \phi_t >_\rho - (\nabla\phi, \nabla\phi)\}dt - \rho\int_0^T(\phi_t, \mathcal{M}\phi_t)dt. \qquad (3.7)
\end{aligned}
$$

Multiplication of the dual system (2.19)–(2.22) by ϕ followed by integration by parts gives

$$
\begin{aligned}
\int\int(\mathcal{C}_\rho\phi_{tt} - \Delta\phi - C)\phi &= C\int\int\phi = 0 \\
&= (\mathcal{C}_\rho\phi_t, \phi)|_0^T - \int\int\phi_t\mathcal{C}_\rho\phi_t + \int\int|\nabla\phi|^2.
\end{aligned}
$$

Therefore

$$
\int_0^T\{< \phi_t, \phi_t >_\rho - (\nabla\phi, \nabla\phi)\}\, dt = (\mathcal{C}_\rho\phi_t, \phi)|_0^T = Y. \qquad (3.8)
$$

Combining (3.7) and (3.8) we obtain

$$\int |\frac{\partial \phi}{\partial \mathbf{n}}|^2 = 2X + C_1 Y + \int\int \{2\nabla\phi \cdot \nabla h \nabla \phi - \rho \phi_t \mathcal{M} \phi_t\}. \tag{3.9}$$

Each of the terms on the right-hand side of (3.9) can be bounded by a multiple of the energy $\mathcal{E}(T) := \|\{\phi^0, \phi^1\}\|_{\mathcal{E}}$. For example, with Y we use conservation of energy and Poincaré's inequality. Let K_1 be such that

$$\int_\omega |\psi|^2 \, d\omega \le K_1 \int_\omega |\nabla\psi|^2 \, d\omega.$$

We obtain

$$\begin{aligned} |Y| &= | < \phi_t(T), \phi(T) >_\rho - < \phi_t(0), \phi(0) >_\rho | \\ &\le < \phi_t, \phi_t >_\rho^{1/2} < \phi, \phi >_\rho^{1/2} \{|_{t=T} + |_{t=0}\} \\ &\le < \phi_t, \phi_t >_\rho^{1/2} (1 + \rho\|\Lambda\|)^{1/2} (\phi, \phi)^{1/2} \{|_{t=T} + |_{t=0}\} \\ &\le \mathcal{E}^{1/2} (K_1 (1 + \rho\|\Lambda\|))^{1/2} (\int_\omega |\nabla\phi|^2 \, d\omega)^{1/2} \{|_{t=T} + |_{t=0}\} \\ &\le 2(K_1(1 + \rho\|\Lambda\|))^{1/2} \mathcal{E}(T) =: C_2 \mathcal{E}(T). \end{aligned} \tag{3.10}$$

For $|X|$ we use that h is bounded on $\bar{\omega}$ and similar estimates to obtain

$$|X| \le C_3 \mathcal{E}(T). \tag{3.11}$$

Since h is C^2 on $\bar{\omega}$ the matrix norm of ∇h is bounded. Consequently there exists C_4 such that

$$\int\int \nabla\phi \cdot \nabla h \nabla\phi \le T C_4 \mathcal{E}(T). \tag{3.12}$$

To estimate the term involving $\mathcal{M}$ we first note that for $\psi \in \tilde{H}_0^1$ we have

$$\begin{aligned} (\psi, \mathcal{M}\psi) &= (\psi, \Lambda(h \cdot \nabla\psi) - h \cdot \nabla\Lambda\psi) \\ &= (h\Lambda\psi, \nabla\psi) - (\psi, h \cdot \nabla\Lambda\psi) \\ &= -(\nabla \cdot (h\Lambda\psi), \psi) - (h \cdot \nabla\Lambda\psi, \psi) \\ &= -(h \cdot \nabla\Lambda\psi + (\nabla \cdot h)\Lambda\psi, \psi) - (h \cdot \nabla\Lambda\psi, \psi) \\ &= -2(h \cdot \nabla\Lambda\psi, \psi) - C_1(\Lambda\psi, \psi) \end{aligned} \tag{3.13}$$

where C_1 is again the constant in Lemma 3.1. By Proposition 2.1 the operator Λ is continuous from $\tilde{L}^2(\omega)$ to $\tilde{H}^1(\omega)/C$ and hence $h \cdot \nabla\Lambda$ is continuous from $\tilde{L}^2(\omega)$ to $L^2(\omega)$. It follows that $\mathcal{M}$ extends to continuous operator on $\tilde{L}^2(\omega)$ (when $\tilde{L}^2(\omega)$ is identified with its dual.) We then have that there exists a C such that

$$\int\int \phi_t \mathcal{M} \phi_t \le C T \mathcal{E}(T). \tag{3.14}$$

Combining (3.10)–(3.14) with (3.9) we obtain that

$$\int |\frac{\partial \phi}{\partial \mathbf{n}}|^2 \le K_1 \mathcal{E}(T) + T K_2 \mathcal{E}(T) = (K_1 + T K_2) \mathcal{E}(0) \tag{3.15}$$

where the constants K_1 and K_2 are independent of T and the initial data. This completes the proof. □

From this estimate one can easily prove the following.

Corollary 3.1 *Suppose* $\{\phi^0, \phi^1\} \in \mathcal{V}$ *then* $\frac{\partial \phi}{\partial \mathbf{n}}$ *satisfies the estimate (3.15). Equivalently, the observation operator* $\Psi : \mathcal{V} \to L^2((0,t)\times\gamma)$ *defined by* $\Psi\{\phi^0, \phi^1\} = \frac{\partial \phi}{\partial \mathbf{n}}|_{\gamma\times(0,t)}$ *extends continuously to* $\mathcal{E}$.

4 Exact Controllability

Consider the homogeneous problem

$$y_{tt} + \rho(\Lambda y_t)_t - \Delta y = C \qquad \text{in } \omega \times \mathbf{R}^+ \tag{4.1}$$

$$y = 0 \qquad \text{on } \gamma \times (0,T) \tag{4.2}$$

$$\{y, y_t\}|_{t=0} = \{y^0, y^1\} \qquad \text{given in } \tilde{H}_0^1(\omega) \times \tilde{L}^2(\omega). \tag{4.3}$$

The dual problem to the control problem (1.5)–(1.8) (equivalently (2.15)–(2.18)) consists of (4.1), (4.2) with terminal data specified in $\tilde{H}_0^1(\omega) \times \tilde{L}^2(\omega)$ together with the the observation

$$z = \frac{\partial y}{\partial \mathbf{n}} \text{ on } \gamma_1. \tag{4.4}$$

Due to the time-reversibility of the problem we may equivalently consider (4.1), (4.2) with the initial data given at time 0 as in (4.3).

The goal is prove exact the observability of (4.1)–(4.4), i.e., that for some $c > 0$ one has

$$\int_0^T \int_{\gamma_1} |z|^2 \, d\gamma \, dt \geq c\mathcal{E}(0). \tag{4.5}$$

We apply the standard multiplier $(\mathbf{x}) \cdot \nabla y$ (see (Lions, 1988)) to solutions of (4.1)–(4.3).

Letting $(\psi, \phi) = \int_\omega \psi\phi \, d\omega$ and denote

$$X = (y_t + \rho\Lambda y_t, \mathbf{x}\cdot\nabla y)|_0^T, \qquad Y = (y_t + \rho\Lambda y_t, y)|_0^T$$

$$\mathcal{M}\psi = \Lambda(\mathbf{x}\cdot\nabla\psi) - \mathbf{x}\cdot\nabla(\Lambda\psi).$$

Using the notation in (3.2) we multiply (4.1) by the standard multiplier $\mathbf{x}\cdot\nabla y$ to obtain

$$\begin{aligned} 0 &= X - \textstyle\int\int\{(y_t + \rho\Lambda y_t)\mathbf{x}\cdot\nabla y_t + (\Delta y)\mathbf{x}\cdot\nabla y\} \\ &= X - T_1 - T_2 \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} T_1 &= \tfrac{1}{2}\textstyle\int\int\{\mathbf{x}\cdot\nabla(y_t)^2 + \rho\mathbf{x}\cdot\nabla(y_t\Lambda y_t) + \rho(\mathbf{x}\cdot\nabla y_t)\Lambda y_t - \rho\mathbf{x}\cdot\nabla(\Lambda y_t)y_t\} \\ &= -\tfrac{n}{2}\textstyle\int\int\{y_t + \rho(\Lambda y_t)y_t\} + \tfrac{\rho}{2}\int\int\{y_t\mathcal{M}y_t\}, \\ T_2 &= \textstyle\int\{\frac{\partial y}{\partial \mathbf{n}}\mathbf{x}\cdot\nabla y\} - \int\int\{\nabla y\cdot(\mathbf{x}\cdot\nabla y)\} \\ &= \textstyle\int \frac{\partial y}{\partial \mathbf{n}}\mathbf{x}\cdot\nabla y - \frac{1}{2}\int \mathbf{x}\cdot\mathbf{n}|\nabla y|^2 + \frac{n-2}{2}\int\int|\nabla y|^2 \\ &= \tfrac{1}{2}\textstyle\int\{\frac{\partial y}{\partial \mathbf{n}}^2\mathbf{x}\cdot\mathbf{n}\} + \frac{n-2}{2}\int\int|\nabla y|^2, \end{aligned}$$

where $\frac{\partial y}{\partial \mathbf{n}}$ denotes the $\partial y/\partial \mathbf{n}$ and otherwise n is the dimension of ω, which is 2 for the present situation. In the calculation of T_2 we have used that $y = 0$ on γ to combine the boundary terms.

Recall the energies:

$$\mathcal{P}(t) = \frac{1}{2}\int_\omega |\nabla w|^2\, d\omega \qquad \mathcal{K}(t) = \frac{1}{2}\int_\omega (w_t + \rho\Lambda w_t) w_t\, d\omega$$

$$\mathcal{E}(t) = \|\{w, w_t\}\|_{\mathcal{E}}^2 = 2(\mathcal{P}(t) + \mathcal{K}(t)).$$

Combining energies with (4.6) we obtain

$$\begin{aligned} 0 \qquad &= X + n \textstyle\int_0^T \mathcal{K}(t)\, dt - (n-2) \int_0^T \mathcal{P}(t)\, dt \\ &= X + (n-1) \textstyle\int_0^T \mathcal{K}(t) - \mathcal{P}(t)\, dt + \frac{1}{2}\int_0^T \mathcal{E}(t)\, dt \\ &\quad - \tfrac{\rho}{2} \textstyle\int\int y_t \mathcal{M} y_t - \frac{1}{2}\int \mathbf{x}\cdot\mathbf{n} |\frac{\partial y}{\partial \mathbf{n}}|^2 \\ &= X + \tfrac{n-1}{2} Y + \tfrac{1}{2} T\mathcal{E}(0) - \tfrac{\rho}{2}\textstyle\int\int \{y_t \mathcal{M} y_t\} - \frac{1}{2}\int \mathbf{x}\cdot\mathbf{n}|\frac{\partial y}{\partial \mathbf{n}}|^2, \end{aligned}$$

where we have used the calculation

$$\begin{aligned} 0 = \int\int (y_{tt} + \rho(\Lambda y)_{tt} - \Delta y + C) y = Y + \int\int (|\nabla y|^2 - y_t(y_t + \rho\Lambda y_t) \\ = Y + 2\int_0^T (\mathcal{P} - \mathcal{K})\, dt. \end{aligned}$$

Next we use the geometrical condition on the control region. On γ_0 we have $\mathbf{x}\cdot\mathbf{n} \leq 0$ and on γ_1 we have $\mathbf{x}\cdot\mathbf{n} \leq R := \max_{\mathbf{x}\in\gamma_1} |\mathbf{x}|$. Therefore we obtain

$$\frac{T}{2}\mathcal{E}(0) \leq \frac{n-1}{2}|Y| + |X| + \frac{\rho}{2}|\int\int y_t \mathcal{M} y_t| + \frac{R}{2}\int_{\gamma_1\times(0,T)} |\frac{\partial y}{\partial \mathbf{n}}|^2\, d\gamma\, dt. \tag{4.7}$$

Let s_1 be the first eigenvalue of $-\mathcal{C}_\rho \Delta_C$ and let λ_1 be the first eigenvalue of the Dirichlet Laplacian on ω. One has that

$$\begin{aligned} s_1 &= \min_{u\in H^2(\omega)\cap \tilde{H}_0^1(\Omega)} \frac{(\nabla u, \nabla u)}{< u, u >_\rho} \geq \min_{u\in H^2(\omega)\cap \tilde{H}_0^1(\Omega)} \frac{(\nabla u, \nabla u)}{\|\mathcal{C}_\rho\| < u, u >_{L^2}} \\ &\geq \min_{u\in H^2(\omega)\cap H_0^1(\omega)} \frac{(\nabla u, \nabla u)}{\|\mathcal{C}_\rho\| < u, u >_{L^2}} = \frac{\lambda_1}{\|C_\rho\|} \end{aligned}$$

Hence one has the following estimate:

$$< u, u >_\rho \leq \frac{\|\mathcal{C}_\rho\|}{\lambda_1}(\nabla u, \nabla u). \tag{4.8}$$

Using (4.8) together with fact that $< \cdot, \cdot >_\rho$ is an inner product gives the estimate

$$\begin{aligned} |(\mathcal{C}_\rho y_t, y)| &\leq \frac{1}{2}(\mathcal{C}_\rho y_t, y_t) + \frac{1}{2}(\mathcal{C}_\rho y, y) \\ &\leq \frac{1}{2}(\mathcal{C}_\rho y_t, y_t) + \frac{\|\mathcal{C}_\rho\|}{2\lambda_1}(\nabla y, \nabla y). \end{aligned} \tag{4.9}$$

Using (4.9) one easily obtains the bound on Y:

$$|Y| \leq \frac{\|\mathcal{C}_\rho\|}{\lambda_1}\mathcal{E}(0). \tag{4.10}$$

Similarly $|X|$ can be bounded in terms of the energy:

$$|X| \leq R\|\mathcal{C}_\rho\|\mathcal{E}(0). \tag{4.11}$$

Finally the term involving $\mathcal{M}$ can be handled in the same way that the corresponding term was handled in (3.13) in the proof of Proposition 3.1. One obtains the estimate

$$|\int\int y_t \mathcal{M} y_t| \leq T(n\|\Lambda\| + RC_\Omega)\mathcal{E}(0) \tag{4.12}$$

where (recall $n = 2$ here) and C_Ω is the constant appearing in Proposition 2.2.

Combining (4.7)–(4.12), we see that

$$\begin{aligned} R\int_0^T \int_{\gamma_1} |\frac{\partial y}{\partial \mathbf{n}}|^2 \, d\gamma \, dt &\geq (T - \rho T(RC_\Omega + n\|\Lambda\|) - (n-1)\frac{\|\mathcal{C}_\rho\|}{\lambda_1} - 2R\|\mathcal{C}_\rho\|)\mathcal{E}(0) \\ &= (T(1 - \rho/\rho_0)) - K)\mathcal{E}(0). \end{aligned} \tag{4.13}$$

where

$$\rho_0 = (RC_\Omega + n\|\Lambda\|)^{-1}, \qquad K = (n-1)\frac{\|\mathcal{C}_\rho\|}{\lambda_1} + 2R\|\mathcal{C}_\rho\|. \tag{4.14}$$

Thus, for $\rho < \rho_0$ the observability estimate (4.5) for (4.1)–(4.4) holds provided that

$$T > \frac{K\rho_0}{\rho_0 - \rho}. \tag{4.15}$$

Finally, the exact controllability of Theorem 1.2 follows by duality. This completes the proof of Theorem 1.2. □

4.1 Concluding Remarks

We have shown that the classical multiplier argument used to establish exact controllability of the wave equation with Dirichlet boundary control can also be applied to the coupled fluid-elastic system (1.5)–(1.8) to obtain the necessary observability estimate mentioned in Remark 1.2, at least provided the fluid density ρ is sufficiently small.

As mentioned in Remark 1.3, it is unknown whether exact controllability holds for all ρ. However, in the sufficient condition that $\rho < \rho_0$, ρ_0 exhibits the same inverse dependence upon the geometric constant C_Ω that was obtained in (Hansen, Lyashenko, 1997) where the moment method was applied (for the case where ω is one dimensional).

It is almost certainly true that the control time T can be improved. In the limit as $\rho \to 0$ the control time obtained here tends to $2R + \frac{n-1}{\lambda_1}$, which is the value obtained in (Lions, 1998) for the wave equation. (This value in turn, can be improved to the optimal value of $2R$ in various ways that do not apply here; see (Lions, 1988).)

Regarding the geometry of Ω, we have taken Ω to be simply connected to obtain the existence of a velocity potential Φ, however, this constraint can be eliminated as in (Hansen, Lyashenko, 1997) by working instead directly with the pressure. Of course the simple connectivity of Ω does not imply that ω need be simply connected. If control is active only on a portion of the boundary γ_1, we have taken $\bar{\gamma}_1 \cap \bar{\gamma}_0 = \emptyset$ to avoid a discussion of singularities. In the case of boundary control of the wave equation various methods have been developed to eliminate the necessity of the geometric condition (1.1), e.g., results on "propogation of singularities"; (see Bardos, et al. 1992).

In the present problem, many properties of hyperbolic partial differential equations e.g., unique continuation properties, finite propogation speed, etc., do not apply here due to the nonlocal nature of the incompressibility constraint. Thus the extent to which the results available for the wave equation apply to the system of this paper is unclear.

Acknowledgments

This research was supported in part by the National Science Foundation under Grant DMS-9623144. In addition a portion was conducted during the author's visit with the Insitut Élie Cartan, Univ. Nancy, France in June of 2000. The author is grateful to Marius Tucsnak (Insitut Élie Cartan) for his valuable discussions and also to the Institut Élie Cartan (June, 2000) for their hospitality and support.

References

Avalos G. (1996): *The exponential stability of coupled hyperbolic/parabolic system arising in structural acoustics.* — Abstract and Applied Analysis **1**, pp.203-219.

Banks H.T., Silcox R.J. and Smith R.C. (1993): *The modeling and control of acoustic/structure interaction problems via peizoceramic actuators: 2-d numerical examples.* — ASME Journal of Vibration and Acoustics **2**, pp.343-390.

Bardos C., Lebeau G., Rauch J. (1992): *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary.* — SIAM J. Control. **30** pp.1024–1065.

Conca C., Osses A., Planchard J. (1998): *Asympotic analysis relating spectral models in fluid-solid vibrations.* — SIAM J. Num. Anal. **35** pp.1020–1048.

Galdi G.D., (1994): *An Introduction to the Mathematical Theory of Navier-Stokes Equations.* Vol. I. — New York: Springer "Tracks in Natural Philosophy" **36**.

Hansen S.W. and Lyashenko A. (1997): *Exact controllability of a beam in an incompressible inviscid fluid.* — Discrete and Cont. Dyn. Sys. Vol 3., pp.59–78.

Lagnese J.E. and Lions J.-L. (1989): *Modeling Analysis and Control of Thin Plates.* — New York: Springer-Verlag, Collection Researches en Mathématiques Appliquées.

Lighthill J. (1981):, *Energy flow in the cochlea.* — J. Fluid Mech., pp.149–213.

Lions J.-L. (1988): *Exact controllability, stabilization and perturbations for distributed systems.* — SIAM Review, **30** pp.1–67.

Lions J.-L., Magenese E. (1972): *Non-Homogeneous Boundary Value Problems and Applications*, vol. I. — New York: Springer-Verlag.

Lions J.-L. and Zuazua E. (1995): *Approximate controllability of hydro-elastic coupled system.* — ESAIM: Control Optim. Calc. Var. (electronic), **1** pp. 1–15.

Micu S. and Zuazua E. (1997): *Boundary controllability of a linear hybrid system arising in the control of noise.* — SIAM J. Contr. Optim. **35**, pp.1614–1637.

Necas J. (1967): *Les Méthodes directes en théorie des équations elliptiques.* — Paris: Masson.

Osses A. (1998): *A rotated direction multiplier technique and some applications: Explicit observability constraints for waves and elasticity and tangential approximate controllability of the stokes system* in: "Quelques méthodes théoriques et numériques de contrôlabilité et problèmes d'interactions fluide-structure" Doctoral Thesis, Ecole Polytechnique.

Osses A. and Puel J.-P. (1999): *Approximate controllability for a linear model of fluid structure interaction.* — ESAIM Control Optim. Calc. Var. (electronic) **4**, pp.497–519.

Pazy A. (1983): *Semigroups of Linear Operators and Applications and Partial Differential Equations.* — New York: Springer-Verlag.