Wan ~~Outb~~ Introduction (Ch. I)

The modeling cycle

- Identify physical problem, parameters, key variables
- Formulate initial math model (simple to start)
- Math, computational analysis
- Validation - data comparisons
- Improved formulation

repeat

Hope to describe
- evolution of system
- stability
- qualitative properties (limit cycles / bifurcations)
- range of validity

add ons:
- feed back
- control
- diffusion (pde)
- ~~limits~~ stochastic terms

population modeling (Wan Ch. 2)

Let $y(t)$ denote population time t
(could be in billions or kg mass depending on problem)
- assume contin. var.

- start with a simple model (later adjust after comparisons with data)

$$\frac{dy}{dt} = f(t,y) \qquad \text{1st order}$$

one hopes that population should depend on y mainly (so to start, assume autonomous).

$$\frac{dy}{dt} = f(y)$$ predict world human population at a later time

properties: if $y = 0$ no growth $\Rightarrow f(0) = 0$

taylor's series $f(y) = f'(0)y + \frac{1}{2}f''(0)y^2 + \dots + \frac{1}{n!}f^{(n)}(0)y^n$

$+ \frac{1}{(n+1)!}f^{(n+1)}(c)\,y^{n+1} \leftarrow$ error term

where $0 < c < y^{n+1}$

for small enough populations

$$f(y) = f'(0)y + \underbrace{\frac{1}{2}f''(c)y^2}_{\text{error}}$$

$$\approx \underbrace{f'(0)}_{a_0}y$$

if $a_0 > 0 \Rightarrow$ growing population

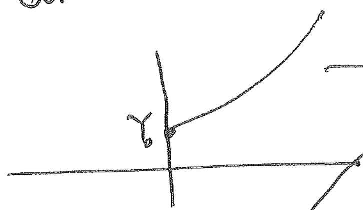
if $a_0 < 0 \Rightarrow$ dying population

$$\underbrace{\frac{dy}{dt} = a_0 y}_{\text{linear growth rate model}} \Rightarrow y = y(0)e^{a_0 t} = Y_0 e^{a_0 t}$$

a_0 = growth rate constant

math analysis of simple population model:

$$\frac{dy}{dt} = a_0 y \qquad y(0) = Y_0 \quad \Rightarrow \quad y = Y_0 e^{a_0 t}$$



If $a_0 > 0$ population grows exponentially

① – might not be realistic due to limited space

② if y is very small, for some types of populations e.g. wolves, it does not match reality
– $a_0 < 0$ with y small enough

$\therefore$ probably valid for limited range of y

$f(y) = a_0 y$:
linear growth rate
a_0: growth rate const.

$$a_0 = \frac{dy/dt}{y} = \text{"relative" growth rate}$$

ex At 10% interest rate, 1000\$ becomes 1100 after 1 year

$$\frac{\Delta y / \Delta t}{y} = \frac{100/1}{1000} = .1$$

Model validation: See how $Y_0 e^{a_0 t}$ fits data

Suppose 2 data points, y: world population at time t

$t = 0$ at 2011

date	t	y
2011	0	7 (billion)
1974	-37	4

$t=0$ $\quad Y_0 e^0 = 7 \qquad Y_0 = 7$

$t=-37$ $\quad Y_0 e^{-37 a_0} = 4$

$$e^{-37 a_0} = 4/7$$

$$a_0 = \frac{-1}{37} \ln\left(\frac{4}{7}\right)$$

Table 2.1 $\left(y = 7e^{a_0 t} \quad a_0 = \frac{-1}{37} \ln\left(\frac{4}{7}\right) \right)$

	1804	1927	1959	1974	1987	1999	2011
t_n	-207	-84	-52	-37	-24	-12	0
$y(t_n)$	.306	1.97	3.19	4	4.87	5.84	7
$\bar{y}_n$ = data	1	2	3	4	5	6	7

(see book)

- 7 % error back to 1927 (1804 was off)

∴ would guess it is accurate for 50 years forward

Model can predict : $y(9) =$ 2020 world pop

$y(39) =$ 2050 " "

when will p = 10 billion ?

Improved parameter estimation

Suppose $y(t) = Y(t, A)$ is the model. $A = \{a_1, a_2, \cdots, a_m\}$ = parameter set

(in above example, $A = \{a_1, a_2\} = \{Y_0, a_0\}$)

Suppose we have data $\bar{y}_k$ at time t_k $k = 1, 2, \cdots, N$

and model values $y_k = Y(t_k, A)$

residual (error) $= E_k = Y(t_k, A) - \bar{y}_k$

Main case of interest: $N > m$

(i) $N = m \Rightarrow$ should be a unique sol.)

$N < m \Rightarrow$ ~~[illegible]~~ many sol's

$N > m$: usually no exact sol.

Therefore minimize errors

Least squares minim. $S_N^2 = \sum_{k=1}^{N} (y_k - \bar{y}_k)^2$

i.e., minimize $\| \{E_k\} \|_{\ell^2}$

$\| \{x_1, \ldots, x_n\} \|_{\ell^2} = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$

Here, $E_k^2 = (y_k - \bar{y}_k)^2 = (Y_0 \underbrace{e^{a_0 t_k}}_{A_k} - \bar{y}_k)^2$

$S_N^2 = \sum_{k=1}^{N} E_k^2$ $\qquad S_N^2$ minimized when $\nabla_A S_N^2 = 0$

$$\frac{\partial S_N^2}{\partial Y_0} = \sum_{k=1}^{N} \frac{\partial E_k^2}{\partial Y_0} = \sum_{k=1}^{N} 2(Y_0 A_k - \bar{y}_k) A_k = 0$$

$$\Rightarrow Y_0 \sum_{k=1}^{N} A_k^2 = \sum_{k=1}^{N} \bar{y}_k A_k \Rightarrow \boxed{Y_0 = \frac{\sum_{k=1}^{N} \bar{y}_k A_k}{\sum_{k=1}^{N} A_k^2}} \quad (1)$$

$$\frac{\partial S_N}{\partial a_0} = \sum_{k=1}^{N} \frac{\partial E_k^2}{\partial a_0} = \sum 2 (Y_0 A_k - \bar{y}_k) Y_0 t_k A_k$$

$$\Rightarrow \boxed{\sum_{k=1}^{N} (Y_0 A_k - \bar{y}_k) t_k A_k = 0} \quad (2)$$

2 eqn's, 2 unknowns. Plug (1) into (2)

(2) of the form $f(a_0) = 0$

- e.g., use bisection method.