How does tempering affect the local and global properties of fractional Brownian motion?

Farzad Sabzikar

Department of Statistics

Iowa State University

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Abstract

In this talk, we discuss the effects of tempering the power law kernel of moving average representation of a fractional Brownian motion (fBm) on some local and global properties of this Gaussian stochastic process. Tempered fractional Brownian motion (TFBM) and tempered fractional Brownian motion of the second kind (TFBMII) are the processes that are considered in order to investigate the role of tempering. Tempering does not change the local properties of fBm including the sample paths and p-variation, but it has a strong impact on the Breuer-Major theorem, asymptotic behavior of the 3rd and 4th cumulants of fBm and the optimal fourth moment theorem.

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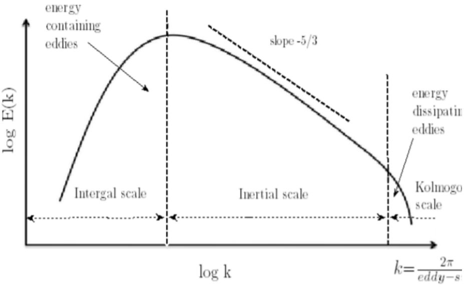
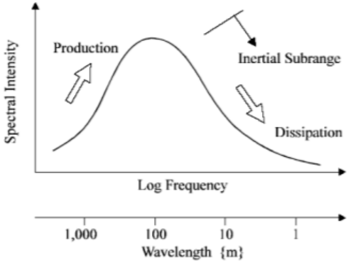


Figure 1.4 Sketch of Kolmogorov energy spectrum for all turbulent flo



(a)

(b)

Figure: Figure reproduced from Beaupuits et al. (2004)

Motivation: Von Kármán model of continuous wind gusts

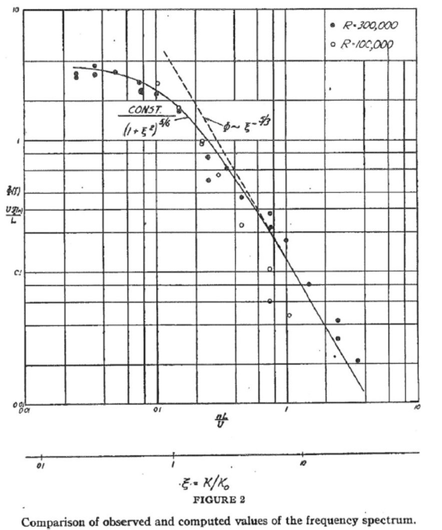


Figure: Von Kármán model of continuous wind gusts. Figure reproduced from Von Kármán (1948)

Long Memory (LM) vs Semi-Long Memory (SLM)

- (LM): A covariance stationary process $\{X_j\}$ with autocovariance function $\gamma(k)$ is said to have long memory if $\sum_{k\in\mathbb{Z}}|\gamma(k)| = \infty$. If $\sum_{k\in\mathbb{Z}}|\gamma(k)| < \infty$ then $\{X_j\}$ has short memory.
- (SLM): A stationary time sires $\{X_j\}$ is called to have semi-long memory if its covariance function which resembles the covariance function of a long memory model for arbitrary large number of lags but eventually decays exponentially fast (Giraitis et al. (2000)).

- The discrete time stochastic process $\{X_t\}_{t\in\mathbb{Z}}$ is called an *autoregressive tempered fractional integrated moving average* time series, denoted by ARTFIMA(p,d,λ,q), if $\{X_t\}_{t\in\mathbb{Z}}$ is a stationary solution with zero mean of the tempered fractional difference equations

$$\Phi(B)X_t = \Theta(B)(1-e^{-\lambda}B)^{-d}Z_t, \tag{0.1}$$

where $BY_t = Y_{t-1}$ is back ward shift operator, $\{Z_t\}_{t\in\mathbb{Z}}$ is a white noise sequence (i.i.d. with $\mathbb{E}[Z_t]=0$ and $\mathbb{E}[Z_t^2]=\sigma^2$), $d\notin\mathbb{Z}$, $\lambda>0$, and $\Phi(z)=1-\phi_1 z-\phi_2 z^2-\ldots-\phi_p z^p$, and $\Theta(z)=1+\theta_1 z+\theta_2 z^2+\ldots+\theta_q z^q$ are polynomials of degrees $p,q\geq 0$ with no common zeros.

- The spectral density of X_t is given by

$$h(\nu) = \frac{\sigma^2}{2\pi} \left| \frac{\Theta(e^{-i\nu})}{\Phi(e^{-i\nu})} \right|^2 \left(1 - 2e^{-\lambda}\cos\nu + e^{-2\lambda}\right)^{-d}, \qquad -\pi \leq \nu \leq \pi.$$

- The covariance function of $X_{0,d,\lambda,0}$ is given by

$$\gamma_{d,\lambda}(k) == \frac{e^{-\lambda k}\Gamma(d+k)}{\Gamma(d)\Gamma(k+1)}\, {}_2F_1(d, k+d; k+1;\, e^{-2\lambda}),$$

where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function. Moreover,

$$\sum_{k\in\mathbb{Z}} |\gamma_{d,\lambda}(k)| < \infty, \qquad \sum_{k\in\mathbb{Z}} \gamma_{d,\lambda}(k) = (1 - e^{-\lambda})^{-2d}$$

and $\gamma_{d,\lambda}(k) \sim Ak^{d-1}e^{-\lambda k}, \quad k \to \infty$, where $A = (1 - e^{-2\lambda})^{-d}\Gamma(d)^{-1}$.

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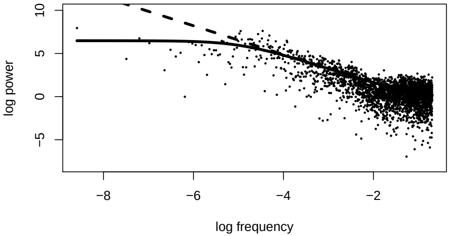
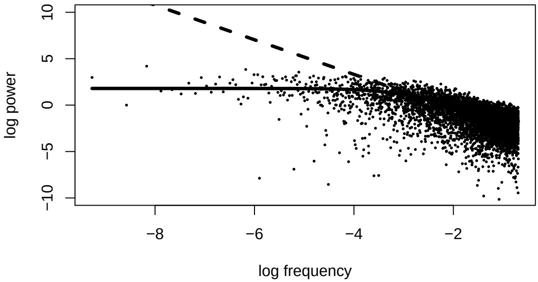


Geophysical turbulence in water velocity data (cm/s) was measured in Lake Michigan, Lake Huron, and the Red Cedar River in Michigan. The figure shows the periodogram and fitted ARTFIMA(p, d, λ, q) spectral density function for a data set from Saginaw Bay. (a) Setting $p = q = 0$ for a tempered fractional noise, we also set $d = 5/6$ (from theory, Kolmogorov scaling and this resulted in the parameter fit $\lambda = 0.045$ (0.00248) using the Whittle estimator.

(b)Without fixing d, the Whittle estimates are $\lambda = 0.027$ $(.00229)$ and $d = 0.752$ $(.00582)$. The ARTFIMA model is stationary, and there is ample reason to consider the time series of velocity data as stationary.



A North American Regional Climate Change Assessment Program (NARCCAP) climate model was used to generate 29 years of daily maximum temperature data at 16,100 spatial locations in North America. We examine this data at one location, with sample size $n = 10,585$. We fit an ARTFIMA$(0, 0.933, 0.3, 0)$ model to the standardized time series. An ARFIMA model fits the data with $d = 0.95$ (not shown). As there is no evidence of nonstationarity in the standardized time series, the stationary ARTFIMA model seems preferable to the ARFIMA model with $d = 0.95$, since the latter has stationary increments.

Functional limit theorem

- A FBM $B_H(t)$ can be obtained as the weak convergence limit of normalized partial sums of an ARFIMA(p, d, q):

$$\frac{1}{N^H}\sum_{k=1}^{[Nt]} X_d(k) \xrightarrow{D[0,1]} \frac{\Theta_q}{\Phi_p \Gamma(d+1)} B_H(t), \text{ as } N \to \infty,$$

where $H = d + \frac{1}{2} \in (0,1)$.

- **Question:** What would be the weak convergence limit of scaled partial sums of ARFIMA(p, d, λ, q) ?
- We answer this question by assuming that the tempering parameter $\lambda \equiv \lambda_N$ may depend on N so that it remains bounded as N increases and following limit exists:

$$\lim_{N\to\infty} N\lambda_N = \lambda_* \in [0, \infty].$$

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- Let $\lambda_* = \infty$, $d \in \mathbb{R} \setminus \mathbb{N}_-$, $\mathbb{E}|\zeta(0)|^p < \infty$ for some $p > 2$. Then

$$N^{-\frac{1}{2}} \lambda_N^d S_N^{d,\lambda_N}(t) \xrightarrow{D[0,1]} \sigma\, B(t),$$

where B is a standard Brownian motion and $\sigma > 0$ some constant.

- Let $\lambda_* = 0$ and $H = d + \frac{1}{2} \in (0,1)$. Moreover, if either $1/2 < H < 1$, or $0 < H < 1/2$ and $\mathbb{E}|\zeta(0)|^p < \infty\,(\exists p > 1/H)$ hold, Then

$$N^{-H} S_N^{d,\lambda_N}(t) \xrightarrow{D[0,1]} \Gamma(d+1)^{-1} B_{H,0}(t),$$

where $B_{H,0} = B_H$ is a multiple of fractional Brownian motion.

- Let $\lambda_* \in (0,\infty)$ and $H = d + \frac{1}{2} > 0$. Assume either $1/2 < H$, or $0 < H < 1/2$ and $\mathbb{E}|\zeta(0)|^p < \infty\,(\exists p > 1/H)$ hold. Then

$$N^{-H} S_N^{d,\lambda_N}(t) \xrightarrow{D[0,1]} \Gamma(d+1)^{-1} B^{II}_{H,\alpha,\lambda_*}(t),$$

where $B^{II}_{H,\alpha,\lambda_*}$ is a stochastic processes which is called tempered fractional Brownian motion of the second kind (TFBM II).

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- Mandelbrot-van-Ness representation of two-sided normalized fractional Brownian motion (fBm) with Hurst index $H \in \left(0, 1\right) \setminus \left\{1/2\right\}$ has a form

$$B_H(t) = C_H \int \left((t-x)_+^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}} \right) B(dx),$$

where $C_H = \frac{(\Gamma(2H+1)\sin(\pi H))^{1/2}}{\Gamma(H+1/2)}$.

TFBM I

- Given an independently scattered Gaussian random measure $B(dx)$ on $\mathbb{R}$ with control measure dx, for any $H > 0$ and $\lambda > 0$, the stochastic process $B^I_{H,\lambda} = \{B^I_{H,\lambda}(t)\}_{t\in\mathbb{R}}$ defined by the Wiener integral

$$B^I_{H,\lambda}(t) := \int \left[e^{-\lambda(t-x)_+}(t-x)_+^{H-\frac{1}{2}} - e^{-\lambda(-x)_+}(-x)_+^{H-\frac{1}{2}} \right] B(dx), \qquad (0.2)$$

where $0^0 = 0$, is called the tempered fractional Brownian motion (TFBM).

- It is easy to check that the function

$$g^I_{H,\lambda,t}(x) := e^{-\lambda(t-x)_+}(t-x)_+^{H-\frac{1}{2}} - e^{-\lambda(-x)_+}(-x)_+^{H-\frac{1}{2}}$$

is square integrable over the entire real line for any $H > 0$, so that TFBM is well-defined. Note that it is defined for $H = 1/2$ as well, in contrast to the Mandelbrot-van-Ness representation, and equals

$$B^I_{1/2,\lambda}(t) = e^{-\lambda t}\int_{-\infty}^{t} e^{\lambda x}\, B(dx) - \int_{-\infty}^{0} e^{\lambda x}\, B(dx).$$

- For $H \neq 1/2$, the kernel $(t-x)_+^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}}$ can be represented as

$$(t-x)_+^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}} = (H-1/2)\int_0^t (s-x)_+^{H-\frac{3}{2}}\,ds.$$

Moderating respectively the integrand by the same exponent, integrating by parts and ignoring normalizing constant, we get another tempered stochastic process: Given an independently scattered Gaussian random measure $B(dx)$ on $\mathbb{R}$ with control measure dx, for any $H > 0$ and $\lambda > 0$, the stochastic process $B^{II}_{H,\lambda} = \{B^{II}_{H,\lambda}(t)\}_{t\in\mathbb{R}}$ defined by the Wiener integral

$$B^{II}_{H,\lambda}(t) := \int g^{II}_{H,\lambda,t}(x)B(dx), \tag{0.3}$$

where

$$g^{II}_{H,\lambda,t}(x) := (t-x)_+^{H-\frac{1}{2}} e^{-\lambda(t-x)_+} - (-x)_+^{H-\frac{1}{2}} e^{-\lambda(-x)_+} + \lambda\int_0^t (s-x)_+^{H-\frac{1}{2}} e^{-\lambda(s-x)_+}\,ds, \quad x\in\mathbb{R}. \tag{0.4}$$

is called the tempered fractional Brownian motion of the second kind (TFBMII).

- TFBM (0.2) and TFBMII (0.3) are Gaussian stochastic processes with stationary increments, having the following scaling property: for any scaling factor $c > 0$

$$\left\{X_{H,\lambda}(ct)\right\}_{t\in\mathbb{R}} \triangleq \left\{c^H X_{H,c\lambda}(t)\right\}_{t\in\mathbb{R}} \tag{0.5}$$

where $X_{H,\lambda}$ could be $B^I_{H,\lambda}$ or $B^{II}_{H,\lambda}$.

- Using the scaling property (0.5) and the fact that $X_{H,\lambda}(|t|)$ has the same distribution as $|t|^H X_{H,\lambda|t|}(1)$, it is easy to see that $\mathbb{E}[(X_{H,\lambda}(|t|))^2] = |t|^{2H}\mathbb{E}[(X_{H,\lambda|t|}(1))^2] =: |t|^{2H}C_t^2$.

- (a) Let $X_{H,\lambda} = B^{I}_{H,\lambda}$. Then the function $C_t^2 = (C_t^I)^2 = \mathbb{E}[(B^{I}_{H,\lambda|t|}(1))^2]$ has the expression
$$(C_t^I)^2 = \frac{2\Gamma(2H)}{(2\lambda|t|)^{2H}} - \frac{2\Gamma(H+\frac{1}{2})}{\sqrt{\pi}}\frac{1}{(2\lambda|t|)^{H}}K_H(\lambda|t|), \tag{0.6}$$
where $t \neq 0$ and $K_\nu(z)$ is the modified Bessel function of the second kind (see Appendix A for the definition of $K_\nu(z)$).

- Let $X_{H,\lambda} = B^{II}_{H,\lambda}$. Then the function $C_t^2 = (C_t^{II})^2 = \mathbb{E}[(B^{II}_{H,\lambda|t|}(1))^2]$ has the expression

$$(C_t^{II})^2 = \frac{(1-2H)\Gamma(H+\frac{1}{2})\Gamma(H)(\lambda t)^{-2H}}{\sqrt{\pi}}\Big[1 - {}_2F_3\big(\{1,-1/2\},\{1-H,1/2,1\},\lambda^2t^2/4\big)$$
$$+ \frac{\Gamma(1-H)\Gamma(H+\frac{1}{2})}{\sqrt{\pi}H2^{2H}}\,{}_2F_3\Big(\{1,H-1/2\},\{1,H+1,H+1/2\},\lambda^2t^2/4\Big), \tag{0.7}$$

where ${}_2F_3$ is the generalized hypergeometric function.

- The TFBM (0.2) with parameters $H > 0$ and $\lambda > 0$ satisfies

$$\lim_{t \to +\infty} \mathbb{E}[B^{I}_{H,\lambda}(t)]^2 = \frac{2\Gamma(2H)}{(2\lambda)^{2H}}. \tag{0.8}$$

- The TFBMII (0.3) with parameters $H > 0$ and $\lambda > 0$ satisfies

$$\lim_{t \to +\infty} \mathbb{E}\left[\frac{B^{II}_{H,\lambda}(t)}{\sqrt{t}}\right]^2 = \lambda^{1-2H}\Gamma^2\left(H + \frac{1}{2}\right). \tag{0.9}$$

- Since TFBM is a Gaussian stochastic process with zero mean, it follows from (0.8) that $B^{I}_{H,\lambda}(t)$ converges in law to a normal random variable with zero mean and variance $2\Gamma(2H)(2\lambda)^{-2H}$ as $t \longrightarrow \infty$, unlike fBm, whose variance diverges to infinity. In contrast, relation (0.9) shows that TFBMII is stochastically unbounded as $t \longrightarrow \infty$.

- Let X stand for be a TFBM $B^{I}_{H,\lambda}$ from (0.2) or for a TFBMII $B^{II}_{H,\lambda}$ from (0.3) both with $0 < H < 1$ and $\lambda > 0$. Then there exist positive constants C_1 and C_2 such that

$$C_1 \left|t-s\right|^{2H} \leq \mathbb{E}\left[\left|X(t)-X(s)\right|^{2}\right] \leq C_2 \left|t-s\right|^{2H} \tag{0.10}$$

for any $s,t \in \left[0,1\right]$.

- Inequalities mean that both processes, TFBM and TFBMII, are quasi-hélices, according to geometric terminology of J.-P. Kahane, see [3].
- The inequality (0.10) holds for any fixed interval $\left[0,T\right]$ with constants C_i depending on T.

- Next, we discuss the existence of local times for TFBM and TFBMII. We also show that these tempered fractional processes are locally nondeterministic on any open interval.
- Suppose $X = \{X(t)\}_{t\geq 0}$ is a real-valued separable random process with Borel sample functions. The random Borel measure

$$\mu_B(A) = \int_B I\{X(s) \in A\}\, ds$$

defined for Borel sets $A \subseteq \mathbb{R},\ B \subseteq \mathbb{R}^+$ is called the occupation measure of X on B. If μ_B is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^+$, then the Radon-Nikodym derivative of μ_B with respect to Lebesgue measure is called the local time of X on B, denoted by $L(B, x)$. See Boufoussi et al. [2] for more detail. For brevity, we denote $L(t, x) := L([0, t], x)$.

- Let X be either TFBM (0.2) or TFBMII (0.3). Then for $0 < H < 1$ and $\lambda > 0$, X has a local time $L(t, x)$ that is continuous in t for a.e. $x \in \mathbb{R}$, and square integrable with respect to x.

- (The idea of the proof): It follows from Boufoussi et al. in [2, Theorem 3.1] that a stochastic process $X = \{X(t)\}_{t\in[0,T]}$ has a local time $L(t, x)$ that is continuous in t for a.e. $x \in \mathbb{R}$, and square integrable with respect to x, if X satisfies the following condition:

  $(\mathscr{H})$: There exist positive numbers $(\rho_0, H) \in (0, \infty) \times (0, 1)$ and a positive function $\psi \in L^1(\mathbb{R})$ such that for all

  $\kappa \in \mathbb{R}, T > 0, t, s \in [0, T], 0 < |t - s| < \rho_0$ we have

$$\left| \mathbb{E} \left[\exp \left(i\kappa \frac{X(t) - X(s)}{|t - s|^H} \right) \right] \right| \leq \psi(\kappa).$$

TFBMI/TFBMII: LND property

- A zero mean Gaussian process $\{X(t)\}_{t\in\mathbb{R}}$ is *locally nondeterministic* (LND) on some interval $\mathbb{T}=(a,b)$ if X satisfies condition (A) consisting of the following three assumptions:

(A) (i) $\mathbb{E}[X^2(t)]>0$ for all $t\in\mathbb{T}$;

(ii) $\mathbb{E}[(X(t)-X(s))^2]>0$ for all $t,s\in\mathbb{T}$ sufficiently close;

(iii) for any $m\geq 2$,

$$\liminf_{\varepsilon\downarrow 0} V_m=\frac{\mathrm{Var}\{X(t_m)-X(t_{m-1})|X(t_1),\ldots,X(t_{m-1})\}}{\mathrm{Var}\{X(t_m)-X(t_{m-1})\}}>0, \qquad (0.11)$$

where the $\liminf$ is taken over distinct, ordered $t_1<t_2<\ldots<t_m\in\mathbb{T}$ with $|t_1-t_m|<\varepsilon$.

- Let X be either TFBM (0.2) or TFBMII (0.3). Then for any $0<H<1$ and $\lambda>0$, X is LND on every interval $(0,T)$ for $0<T<\infty$.

- Fix a time interval $[a,b] \subset \mathbb{R}$, and consider the uniform partition

$$\pi^n = \{a = t_0^n < t_1^n < \ldots < t_n^n = b\},$$

where $t_i^n = a + \frac{i}{n}(b-a)$ for $i = 0, \ldots, n$. Let $\beta \geq 1$ and $X = \{X_t, t \in \mathbb{R}\}$ be a continuous stochastic process. Moreover, we define $\Delta_i^n X = X(t_i^n) - X(t_{i-1}^n)$.

- For any $\beta \geq 1$ the β-variation of X on the interval $[a,b]$, denoted by $\langle X \rangle_{\beta,[a,b]}$, is the limit in probability of

$$S_{\beta,n}^{[a,b]}(X) := \sum_{i=1}^{n} |\Delta_i^n X|^{\beta},$$

if the limit exists. We say that the β-variation of X on $[a,b]$ exists in L^p if the above limit exists in L^p for some $p \geq 1$.

- Let X be either a TFBM $B_{H,\lambda}$ with parameters $H \in (0,1)$ and $\lambda > 0$, defined by (0.2) or a TFBMII $B^{II}_{H,\lambda}$ given by (0.3). Then

$$\langle B_{H,\lambda} \rangle_{\frac{1}{H},[a,b]} = c_H(b-a)$$

in probability, where $c_H = C(H)^{-\frac{1}{H}} \mathbb{E}\left[|Z|^{\frac{1}{H}}\right]$ and Z is a $\mathcal{N}(0,1)$-random variable.

- The p-variation of a TFBM and a TFBMII equals zero or infinity, depending on whether p is greater or less than $1/H$.

Tempered fractional Gaussian noises

- Denote $\alpha = H - \frac{1}{2}$. Given a TFBM (0.2), we define tempered fractional Gaussian noise (TFGN) $X^{I}_{\alpha,\lambda}(j) = B^{I}_{H,\lambda}(j+1) - B^{I}_{H,\lambda}(j)$ for $j \in \mathbb{Z}$. It follows easily from (0.2) that TFGN has the moving average representation:

$$\begin{aligned} X^{I}_{\alpha,\lambda}(j) &= \int_{\mathbb{R}} g^{I}_{\lambda,\alpha,j}(x) B(dx) \\ &= \int_{\mathbb{R}} \left[e^{-\lambda(j+1-x)_+}(j+1-x)^{\alpha}_{+} - e^{-\lambda(j-x)_+}(j-x)^{\alpha}_{+} \right] B(dx). \end{aligned} \tag{0.12}$$

- A tempered fractional Gaussian noise of the second kind (TFGNII) can be defined as $X^{II}_{\alpha,\lambda}(j) = B^{II}_{H,\lambda}(j+1) - B^{II}_{H,\lambda}(j)$ for $j \in \mathbb{Z}$. It follows from (0.4) that a TFGNII has the moving average representation

$$\begin{aligned} X^{II}_{\alpha,\lambda}(j) = \int_{\mathbb{R}} g^{II}_{\lambda,\alpha,j}(x) B(dx) = \int_{\mathbb{R}} \Big[& e^{-\lambda(j+1-x)_+}(j+1-x)^{\alpha}_{+} - e^{-\lambda(j-x)_+}(j-x)^{\alpha}_{+} \\ & + \lambda \int_{j}^{j+1} e^{-\lambda(s-x)_+}(s-x)^{\alpha}_{+} ds \Big] B(dx). \end{aligned} \tag{0.13}$$

- let $X^{I}_{\alpha,\lambda}(j)$ and $X^{II}_{\alpha,\lambda}(j)$ be the stationary sequences given by (0.12) and (0.13) respectively. Denote

$$\gamma^{J}(k) := \mathbb{E}[X^{J}_{\alpha,\lambda}(0)X^{J}_{\alpha,\lambda}(k)]$$
$$= |k+1|^{2H}(C^{J}_{|k|+1})^2 - 2|k|^{2H}(C^{J}_{|k|})^2 + |k-1|^{2H}(C^{J}_{|k-1|})^2, \; J = I, II, \tag{0.14}$$

where the normalizing constants C^{J}_{t} are presented as before.

- Let $\lambda > 0$. TFGN is negatively correlated when $H \in \left(0, \frac{1}{2}\right]$ meaning that for every $0 \neq k \in \mathbb{Z}$

$$\gamma^{I}(k) < 0. \tag{0.15}$$

- Let $\lambda > 0$. Then for every $k \in \mathbb{Z}$ and $H > 1/2$,

$$\gamma^{II}(k) > 0. \tag{0.16}$$

Moreover, when $H = 1/2$, it holds $\gamma^{II}(0) > 0$, and $\gamma^{II}(k) = 0$ for every $0 \neq k \in \mathbb{Z}$.

Asymptotic behavior of covariances of TFGNI/TFGNII

- For any $\alpha > -\frac{1}{2}$,

$$\gamma^{I}(j) \sim -\frac{2\Gamma(\alpha+1)(\cosh\lambda - 1)}{(2\lambda)^{\alpha+1}} e^{-\lambda j} j^{\alpha} \tag{0.17}$$

as $j \to \infty$. It means that asymptotically TFGN has negative correlation for any $\alpha > -\frac{1}{2}$. In particular, $\gamma^{I} \in \ell^{q}(\mathbb{Z})$ for every $q \geq 1$.

- For any $\alpha > -\frac{1}{2}$,

$$\gamma^{II}(j) \sim (2e^{\lambda} - 1)(2\lambda)^{-\alpha-1}\Gamma(\alpha+1)e^{-\lambda j} j^{\alpha-1}$$

as $j \to \infty$. It means that asymptotically TFGNII has positive correlation. In particular, $\gamma^{II} \in \ell^{q}(\mathbb{Z})$ for every $q \geq 1$.

Asymptotic behavior of covariances of TFGNI/TFGNII

- Let $Y^{I}(j) = \left(C_1^{I}\right)^{-1} X_{\alpha,\lambda}^{I}(j)$ and $Y^{II}(j) = \left(C_1^{II}\right)^{-1} X_{\alpha,\lambda}^{II}(j)$ be normalized tempered fractional Gaussian noises with associated normalizing constants C_1^{I} and C_1^{II}. Let $V_{n,q}^{J} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} H_q\left(Y^{J}(k)\right)$, $J = I, II$, where H_q stands for the qth Hermite polynomial. Then

$$\begin{aligned} \sigma_{n,J,q}^2 &:= \mathrm{Var}\left(V_{n,q}^{J}\right) = \frac{q!}{n} \left(C_1^{J}\right)^{-2q} \sum_{k,l=1}^{n} \left(\gamma^{J}(k-l)\right)^q \\ &\longrightarrow \sigma_{J,q,H,\lambda}^2 := q! \left(C_1^{J}\right)^{-2q} \sum_{k\in\mathbb{Z}} \left(\gamma^{J}(k)\right)^q < +\infty. \end{aligned} \tag{0.18}$$

Furthermore we can guarantee this value is strictly positive provided that

(a) $J = I, II$ and q is even.

(b) $J = I$, $H \in (0, 1/2]$ and $q > 1$.

(c) $J = II$, $H \geq 1/2$ and $q > 1$.

- Let $\gamma(dx) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ denote the standard Gaussian measure on the real line. Assume that $f \in L^2(\mathbb{R}, \gamma)$ be a centered function, i.e. $\mathbb{E}_\gamma[f] = 0$, with Hermite rank $d \geq 1$, meaning that, f admits the Hermite expansion $f(x) = \sum_{j=d}^{\infty} a_j H_j(x)$ with $a_d \neq 0$. We have that

$$V^J_{n,d,H,\lambda} := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(Y^J_j) \xrightarrow{\ \mathrm{d}\ } \mathcal{N}(0, \sigma^2_{J,H,\lambda,d}),$$

with

$$\sigma^2_{J,H,\lambda,d} = \sum_{q=d}^{\infty} \frac{q!}{2^q} a_q^2 \sigma^2_{J,q,H,\lambda} \in [0, \infty), \tag{0.19}$$

where $\sigma^2_{J,q,H,\lambda}$ is introduced in (0.18).

In any of the following cases: (a) $J = I, II$ and $a_q \neq 0$ for at least one of even q; (b) $J = I$ and $H \leq 1/2$; (c) $J = II$ and $H \geq 1/2$ we claim that $\sigma^2_{J,H,\lambda,d} > 0$.

- The message of last theorem is that tempering always fulfills the sufficient condition in the Breuer-Major Theorem without assuming any extra condition on the Hurst parameter H or/and the tempering parameter λ. This is in contract to the classical setup of the fractional Gaussian noise where often there is a phase transition for the validity of CLT, see [25, Theorem 7.4.1].

A quantitative version of the CLTs for TFGNI/TFGNII

- For given random elements F and G the *total variation* distance, denoted by d_{TV}, between the laws of F and G defined as

$$d_{TV}(F,G) := \sup_A \Big| \mathbb{P}(F \in A) - \mathbb{P}(G \in A) \Big|$$

where the supremum is taken over all the Borel subsets $A \in \mathscr{B}(\mathbb{R})$ on the real line.

- We introduce the Sobolev space $\mathbb{D}^{p,k}(\mathbb{R},\gamma)$, where $p \geq 1$ and $k \in \mathbb{N}$, that is the closure of all polynomial mapping $f : \mathbb{R} \to \mathbb{R}$ with respect to the norm

$$\|f\|_{p,k} := \left[\sum_{i=0}^{k} \int_{\mathbb{R}} |f^{(i)}(x)|^p \gamma(dx) \right]^{\frac{1}{p}}.$$

Here $f^{(0)} = f$, and $f^{(i)}$ stands for the ith derivative of f, $i = 1, \ldots, k$.

- Let the random variable $N \sim \mathcal{N}(0,1)$, $\gamma(dx) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ denote the standard Gaussian measure on the real line. Assume that $f \in L^2(\mathbb{R}, \gamma)$ be a centered function, i.e. $\mathbb{E}_\gamma[f] = 0$, with Hermite rank $d \geq 1$, meaning that, f admits the Hermite expansion $f(x) = \sum_{j=d}^{\infty} a_j H_j(x)$ with $a_d \neq 0$, and $\sigma^2_{J,H,\lambda,d} > 0$. If $f \in L^2(\mathbb{R}, \gamma)$ with $\mathbb{E}_\gamma[f] = 0$ and belongs to Sobolev space $\mathbb{D}^{1,4}(\mathbb{R}, \gamma)$, then

$$d_{TV}\left(\frac{V_n^J}{\sqrt{\mathrm{Var}\left(V_n^J\right)}}, N\right) = \mathscr{O}\left(n^{-\frac{1}{2}}\left(\sum_{|v| \leq n} \left|\gamma^J(v)\right|\right)^{\frac{3}{2}}\right), \; n \to \infty. \tag{0.20}$$

So, $d_{TV}\left(\frac{V_n^J}{\sqrt{\mathrm{Var}(V_n^J)}}, N\right) \leq C\, n^{-\frac{1}{2}}$ for some constant C, and $n \geq 1$. Here $J = I, II$.

Asymptotic behavior of the third and fourth cumulants of TFGNI/TFGNII

- Let F be a real-valued random variable with $\mathbb{E}|F|^n < \infty$ for $n \geq 1$. Let $\phi_F(t) = \mathbb{E}[e^{itF}]$ be the characteristic function of F. Then

$$\kappa_j(F) = (-i)^j \frac{d^j}{dt^j} \log \phi_F(t)|_{t=0}$$

is called the jth cumulant of F. For every $n \geq 1$, define

$$F_n^J = \frac{V_n^J}{\sqrt{\mathrm{Var}(V_n^J)}} = \frac{1}{\sqrt{n\mathrm{Var}(V_n^J)}} \sum_{k=1}^{n} H_q(Y^J(k)), \quad J = I, II. \tag{0.21}$$

- Let $q \geq 2$ be an integer. Consider sequence $\left(F_n^J : n \geq 1\right)$ given by relation (0.21) where H_q denote the Hermite polynomial of degree q. Then, as n tends to infinity,
  (a) For any even integer $q \geq 2$, it holds that $\kappa_3\left(F_n^J\right) \asymp n^{-\frac{1}{2}}$.
  (b) For any integer $q \geq 2$, it holds that $\kappa_4\left(F_n^J\right) \asymp n^{-1}$ provided that either q is even, or $J = I$, $H \in \left(0, 1/2\right]$, or $J = II$, $H \geq 1/2$.

Therefore, if $q \geq 2$ is an even integer, then there exist two constants $C_1, C_2 > 0$ (independent of n) so that for every $n \geq 1$, the following optimal third moment estimate holds:

$$C_2 \left|\mathbb{E}\left[\left(F_n^J\right)^3\right]\right| \leq d_{TV}\left(F_n^J, N\right) \leq C_1 \left|\mathbb{E}\left[\left(F_n^J\right)^3\right]\right|. \tag{0.22}$$

- The tempering parameter λ manifests its role in the optimal fourth moment theorem. In fact, the optimal rates of convergence of the third and fourth cumulants of F_n^J are valid for any $H > 0$ and $\lambda > 0$ for even q. This is in contrast with the case of fractional Brownian motion where $\kappa_3(F_n) \asymp n^{-\frac{1}{2}}$ provided $H \in \left(0, 1 - \frac{2}{3q}\right)$ with an even integer $q \geq 2$ and $\kappa_4(F_n) \asymp n^{-1}$ provided $H \in \left(0, 1 - \frac{3}{4q}\right)$ with $q \in 2, 3$, see Propositions 6.6 and 6.7 in [1].
- It is also worth to mention that for q even the sequence $\left(F_n^J : n \geq 1\right)$ given by (0.21) exhibits the interesting scenario that $\kappa_3(F_n^J) \approx \left(\kappa_4(F_n^J)\right)^{\frac{1}{2}}$, and hence the third cumulant $\kappa_3(F_n^J)$ asymptotically dominates the fourth cumulant as n tends to infinity. A similar phenomenon appears in [4] as well, in which, convergence of third cumulants to zero implies the convergence of the fourth cumulants to zero.

Summary

- TFBMI/TFBMII
- Sample paths and LND properties
- *p*-variation
- TFGNI/TFGNII
- Breuer–Major theorem for TFGNI/TFGNII
- Optimal fourth moment for TFGNI/TFGNII

H. Biermé, A. Bonami, and I. Nourdin.

Optimal berry-esseen rates on the wiener space: the barrier of third and fourth cumulants.

ALEA Lat. Am. J. Probab. Math. Stat. 9, pages 473–500, 2012.

B. Boufoussi, M. E. Dozzi, and R. Guerbaz.

Path properties of a class of locally asymptotically self similar processes.

Electron. J. Probab., 13:898–921, 2008.

J. P. Kahane.

Hélices et quasi-hélices.

Academic Press, New York, NY, 1981.

L. Neufcourt and F. G. Viens.

A third-moment theorem and precise asymptotics for variations of stationary gaussian sequences.

ALEA Lat. Am. J. Probab. Math. Stat., 13:239–264, 2016.

1. J. P. Pérez Beaupuits, A. Otárola, F. T. Rantakyrö, R. C. Rivera, S. J. E. Radford, and L.-Å. Nyman (2004) Analysis of wind data gathered at Chajnantor. *ALMA Memo* **497**.
2. Giraitis, L., Kokoszka, P. and Leipus, R. (2000). Stationary ARCH models: dependence structure and central limit theorem. *Econometric Th.* **16** 3–22.
3. Giraitis, L., Kokoszka, P., Leipus, R. and Teyssière, G. (2003). Rescaled variance and related tests for long memory in volatility and levels. *J. Econometrics* **112** 265–294.
4. Giraitis, L., Leipus, R. and Surgailis, D. ARCH(∞) models and long memory properties. (2009). In: Andersen, T.G., Davis, R.A., Kreiss, J.-P. and Mikosch, T. (eds.) *Handbook of Financial Time Series*, pp. 71–84. New York: Springer-Verlag.
5. Andrej N. Kolmogorov, A. N. (1941) The Local Structure of Turbulence in Incompressible Viscous Fluid for Very Large Reynolds' Numbers. Akademiia Nauk SSSR Doklady, 30:301305.

6. Giraitis, L., Koul. H.L. and Surgailis, D. (2012). *Large Sample Inference for Long Memory Processes.* London: Imperial College Press.
7. Kokoszka, P.S. and Taqqu, M.S. (1995). Fractional ARIMA with stable innovations. *Stoch. Process. Appl.*, **60**, 19–47.
8. Meerschaert, M.M., Sabzikar, F. (2013). Tempered fractional Brownian motion, Statistics and Probability Letters, Vol. 83, No. 10, pp. 2269–2275(2013).
9. Sabzikar, F., Surgailis, D.: Tempered fractional Brownian and stable motions of second kind, *Statistics and Probability Letters*, Vol. 132 (2017), pp. 17-27.
10. Sabzikar, F., Surgailis, D.: Invariance Principles for Tempered Fractionally Integrated Processes, *Stochastic Processes and their Applications*, Vol. 128 (2018), No. 10, pp. 3419–3438.
11. Von Kármán, T. (1948). Progress in the statistical theory of turbulence. *Proc Natl Acad Sci USA*, **34**, 530–539

Thank You!

- A tempered fractional stable motion, TFSM, is a stochastic processes defined by

$$Z_{H,\alpha,\lambda}(t) = \int_{\mathbb{R}} \left((t-y)_+^{H-\frac{1}{\alpha}} e^{-\lambda(t-y)_+} - (-y)_+^{H-\frac{1}{\alpha}} e^{-\lambda(-y)_+} \right) M_\alpha(dy),$$

for $0 < \alpha \leq 2, 0 < H < 1, \lambda > 0$ and symmetric α-stable Lévy process M_α.

- For $\lambda = 0$ both TFSM and TFSM II agree with fractional stable motion.

- For any $\lambda > 0$ and $\kappa > 0$, we define the (positive and negative) tempered fractional integrals (TFI) of a function f by

$$\mathbb{I}_{\pm}^{\kappa,\lambda} f(y) = \frac{1}{\Gamma(\kappa)} \int_{\mathbb{R}} f(u)(u-y)_{\pm}^{\kappa-1} e^{-\lambda(u-y)_{\pm}} du.$$

- For $0 < \kappa < 1$ and $\lambda > 0$, we define the (positive and negative) tempered fractional derivatives (TFD) of a function f by

$$\mathbb{D}_{\pm}^{\kappa,\lambda} f(y) = \lambda^{\kappa} f(y) + \frac{\kappa}{\Gamma(1-\kappa)} \int_{\mathbb{R}} \frac{f(y)-f(u)}{(y-u)_{\pm}^{\kappa+1}} e^{-\lambda(y-u)_{\pm}} du.$$

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- The variance and covariance of TFBM II $B^{II}_{H,\lambda}$ $(H > 0, \lambda > 0)$ has the form

$$C_t^2 = \mathbb{E}\left[\left(B^{II}_{H,\lambda}(t)\right)^2\right] = \frac{1}{2\pi}\int_{\mathbb{R}}\left|\frac{e^{\mathrm{i}\omega t}-1}{\mathrm{i}\omega}(\lambda+\mathrm{i}\omega)^{\frac{1}{2}-H}\right|^2 d\omega$$
$$= \frac{\Gamma(-1/2)\Gamma(H)}{\pi\Gamma(H-\frac{1}{2})\lambda^{2H}}\left[1 - {}_2F_3\left(\{1,-1/2\},\{1-H,1/2,1\},\frac{\lambda^2 t^2}{4}\right)\right]$$
$$+ 2^{1-2H}\pi^{-1}\Gamma(H)t^{2H}{}_2F_3\left(\{1,H-\frac{1}{2}\},\{1,H+1,H+\frac{1}{2}\},\frac{\lambda^2 t^2}{4}\right), \tag{0.23}$$

and

$$\mathrm{Cov}\left[B^{II}_{H,\lambda}(t), B^{II}_{H,\lambda}(s)\right] = \frac{1}{2}\left[C_t^2 + C_s^2 - C_{t-s}^2\right], \qquad s,t\in\mathbb{R}, \tag{0.24}$$

where C_t^2 is given in (0.23) and ${}_2F_3$ is the generalized hypergeometric function.

- We can extend the definition of tempered fractional derivatives to a suitable class of functions in $L^2(\mathbb{R})$. For any $\kappa > 0$ and $\lambda > 0$ we may define the fractional Sobolev space

$$W^{\kappa,2}(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (\lambda^2 + \omega^2)^{\kappa} |\hat{f}(\omega)|^2 \, d\omega < \infty\},$$

which is a Banach space with norm $\|f\|_{\kappa,\lambda} = \|(\lambda^2 + \omega^2)^{\kappa/2} \hat{f}(\omega)\|_2$.

- Let $\kappa > 0$ and $f \in L_1(\mathbb{R})$ (or $L_2(\mathbb{R})$). Then $\mathbb{I}_{-}^{\kappa,\lambda} f(t)$ has the Fourier transform

$$\widehat{\mathbb{I}_{\pm}^{\kappa,\lambda} f}(\omega) = \hat{f}(\omega)(\lambda \pm i\omega)^{-\kappa}$$

for $\lambda > 0$. For $f \in W^{\kappa,2}(\mathbb{R})$,

$$\widehat{\mathbb{D}_{\pm}^{\kappa,\lambda} f}(\omega) = \hat{f}(\omega)(\lambda \pm i\omega)^{\kappa}$$

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