Presentations and theories of metric structures

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SEALS 2022

Convention

$\mathbb{F}$ = the field of scalars $\in \{\mathbb{R}, \mathbb{C}\}$ (usually $\mathbb{C}$).

Definition

A *metric structure* consists of the following.

1. A non-empty set X (the domain of the structure).
2. A complete metric on X.
3. For each $n \in \mathbb{N}$, a possibly empty set of uniformly continuous n-ary operations on X (i.e. functions from X^n into X).
4. For each $n \in \mathbb{N}$, a possibly empty set of uniformly continuous $(n+1)$-ary functionals on X (i.e. functions from X^{n+1} into $\mathbb{F}$).

We treat constants as 0-ary operations.

The following are (or can be represented as) metric structures.

1. Complete metric spaces.
2. Banach spaces. (One unary scalar-multiplication operator per scalar.)
3. Hilbert spaces.
4. Probability spaces.
5. C^* algebras.
6. Every countable algebraic structure (use discrete metric).

In continuous logic, structures are bounded. We will introduce this assumption later.

Definition

Suppose $\mathcal{M}$ is a metric structure and $A \subseteq \text{dom}(\mathcal{M})$. The *algebra generated by A* is the smallest subset of $\text{dom}(\mathcal{M})$ that is closed under every operation of $\mathcal{M}$ and includes A.

Definition

$(\mathcal{M}, (d_n)_{n\in\mathbb{N}})$ is a *presentation* of $\mathcal{M}$ if the algebra generated by $\{d_0, d_1, \ldots\}$ is dense in $\mathcal{M}$. d_n is called the n-th *distinguished point* of the presentation.

Definition

Each point in the algebra generated by the distinguished points of $\mathcal{M}^\#$ is called a *rational point* of $\mathcal{M}^\#$. We denote the set of rational points of $\mathcal{M}^\#$ by $\mathbb{Q}(\mathcal{M}^\#)$.

Examples of presentations:

1. $\mathbb{F}$; take the n-th distinguished point to be 1. Rational points $=$
$$\begin{cases} \mathbb{Q} & \mathbb{F} = \mathbb{R} \\ \mathbb{Q}(i) & \mathbb{F} = \mathbb{C}. \end{cases}$$
2. $C[0,1]$; take the n-th distinguished point to be $x \mapsto x^n$. Rational points $=$ rational polynomials.
3. When $p \geq 1$ is a computable real, $L^p[0,1]$; take the n-th distinguished point to be the indicator function of the n-th rational subinterval of $[0,1]$ (in some effective enumeration). Rational points $=$ "rational step functions".

Remark

Distinguished points may not be dense, but the rational points are.

Remark

A countable dense subset of dom($\mathcal{M}$) together with the metric of $\mathcal{M}$ specifies all the points of $\mathcal{M}$. The rational points of a presentation specify the points of $\mathcal{M}$ and the operations and functionals of $\mathcal{M}$.

Convention

1. From now on we will assume our structures have countably many operations and functionals. (In the case of Banach spaces, it suffices to consider $\times$ by rational scalars only.)
2. Furthermore, we will assume the operations and functionals are computably numbered. That is, from the number assigned to ϕ we can compute the arity of ϕ and a modulus of continuity of ϕ.

With this convention, we can represent rational points by finite words. Each presentation then induces a related set of computable points.

Definition

Suppose $\mathcal{M}^\#$ is a presentation of a metric structure and p is a point of $\mathcal{M}$. p is a *computable point of* $\mathcal{M}^\#$ if there is an algorithm that given $k \in \mathbb{N}$ as input computes a rational point q of $\mathcal{M}^\#$ so that $d(p, q) < 2^{-k}$.

Definition

A presentation $\mathcal{M}^{\#}$ of a metric structure is *computable* if the metric and functionals are computable on the rational points of $\mathcal{M}^{\#}$.

Definition

$\mathcal{M}^{\#}$ is *computably compact* if it is computable and if there is an algorithm that given any $k \in \mathbb{N}$ produces a nonempty and finite $F \subseteq \mathbb{Q}(\mathcal{M}^{\#})$ so that $d(p, F) < \epsilon$ for every point p of $\mathcal{M}$.

Theorem (Specker/Folklore)

If $\mathcal{M}^{\#}$ is computably compact, and if $f : \mathcal{M}^{\#} \to \mathbb{R}$ is computable, then the maximum value of f is computable.

Theorem (Stone-Weierstrass Theorem)

*Suppose X is a compact metric space and $S \subseteq C(X)$. If S separates points, then the unital *-algebra generated by S is dense in $C(X)$.*

Theorem (M. 2019(?)+)

If X has a computably compact presentation, then $C(X)$ and $C^*(X)$ are computably presentable.

Proof: Let $X^\#$ be a computably compact presentation of X. Let:

$$\begin{aligned} p_n &= n\text{th distinguished point of } X^\# \\ g_0(p) &= 1 \ \forall p \in X \\ g_{n+1} &= d_X(p, p_n) \ \forall p \in X \\ C^*(X)^\# &= (C^*(X), (g_n)_{n\in\mathbb{N}}) \\ \mathcal{A} &= \text{subalgebra of } C^*(X) \text{ generated by } \{g_n \ : \ n \in \mathbb{N}\} \end{aligned}$$

Claim 1: $C^*(X)^\#$ is a presentation of $C^*(X)$.

Proof Claim 1: STS $\mathcal{A}$ is dense in $C^*(X)$. Suppose $p, q \in X$ are distinct. Since $\{p_n \ : \ n \in \mathbb{N}\}$ is dense in X, there exists $n \in \mathbb{N}$ so that $g_n(p) \neq g_n(q)$. That is, $\{g_n \ : \ g \in \mathbb{N}\}$ separates points. By the Stone-Weierstrass Theorem, $\mathcal{A}$ is dense in $C(X)$. Since $C(X)$ and $C^*(X)$ have the same points, $\mathcal{A}$ is dense in $C^*(X)$. $\square_{\text{Claim 1}}$

Note that $\mathcal{A} =$ the set of rational points of $C^*(X)^\#$.

For each finite $F \subseteq \mathbb{N}$, set

$$h_F = \prod_{j \in F} g_j.$$

Claim 2: $C^*(X)^\#$ is computable.

Proof Claim 2: STS $\| \ \|_{\sup}$ is computable on $\mathcal{A}$. Let $f \in \mathcal{A}$. From a code of f we may compute $n \in \mathbb{N}$, $\alpha_0, \ldots, \alpha_m \in \mathbb{Q}(i)$ and $F_0, \ldots F_m \in \mathrm{Fin}(\mathbb{N})$ so that

$$f = \sum_j \alpha_j h_{F_j}.$$

Each g_j is a computable function from $X^\#$ into $\mathbb{R}$ uniformly in j. Thus, f is a computable function from $X^\#$ into $\mathbb{R}$. Since $X^\#$ is computably compact, $\|f\|_{\sup}$ is computable uniformly in a code f. $\square$Claim 2

When $n \in \mathbb{N}$ is a canonical index of $F \in \mathrm{Fin}(\mathbb{N})$, let $f_n = h_F$.

Let $C(X)^\# = (C(X), (f_n)_{n \in \mathbb{N}})$.

Claim 3: $C(X)^{\#}$ is a presentation of $C(X)$.

Proof of Claim 3: $\mathcal{A} =$ subalgebra of $C(X)$ generated by $\{f_n : n \in \mathbb{N}\}$. Therefore, by Claim 1, $C(X)^{\#}$ is a presentation of $C(X)$. $\square_{\text{Claim 3}}$

Claim 4: $C(X)^{\#}$ is a computable presentation of $C(X)$.

Proof of Claim 4: By Claim 2. $\square_{\text{Claim 4}}$ $\square$

Corollary

The following are computably presentable.

1. $C(2^\omega)$ *(Goldring-Hart 2020)*
2. $C(\mathbb{P})$ *(Goldbring-Hart 2020)*
3. $C^*(2^\omega)$ *and* $C^*(\mathbb{P})$.
4. $C(\hat{B})$, *when* B *is a computably presentable Boolean algebra. (Bazhenov, Harrison-Trainor, Melnikov 2021+)*

Translation to continuous logic

Changes in vocabulary:

- 'Functionals' $\rightarrow$ 'predicates'
- 'Operators' $\rightarrow$ 'functions'

Additional assumptions:

- Metric structures bounded (by 1).
- Predicates map into $[0, 1]$ ($0 =$ truth and $1 =$ falsehood).

- '$\forall$' $\rightarrow$ 'sup'
- '$\exists$' $\rightarrow$ 'inf'
- '$=$' $\rightarrow$ 'd'
- Connectives $\neg$, $\frac{1}{2}\times$, and $\dot{-}$
- Language $=$ signature of a metric structure. *Includes a modulus of continuity for each function and predicate symbol.*
- Fix a computably numbered language L (moduli of cont. computable uniformly). Assume all metric structures are interpretations of L.
- $\phi^{\mathcal{M}} =$ truth value of ϕ in metric structure $\mathcal{M}$.

Theorem (CGM 2021+)

Let $\mathcal{M}$ be a computably presentable L-structure.

1. $\{(\phi, r) : \phi$ *q.f.* $\wedge\ \phi^{\mathcal{M}} < r \in \mathbb{Q}^+\}$ *is* Σ^0_1.
2. $\{(\phi, r) : \phi$ *q.f.* $\wedge\ \phi^{\mathcal{M}} \leq r \in \mathbb{Q}^+\}$ *is* Π^0_1.
3. *For every positive integer N:*
   1. $\{(\phi, r) : \phi \in \Pi_N \wedge \phi^{\mathcal{M}} \leq r \in \mathbb{Q}^+\}$ *is* Π^0_N.
   2. $\{(\phi, r) : \phi \in \Pi_N \wedge \phi^{\mathcal{M}} < r \in \mathbb{Q}^+\}$ *is* Σ^0_{N+1}.
   3. $\{(\phi, r) : \phi \in \Sigma_N \wedge \phi^{\mathcal{M}} \leq r \in \mathbb{Q}^+\}$ *is* Σ^0_{N+1}.
   4. $\{(\phi, r) : \phi \in \Sigma_N \wedge \phi^{\mathcal{M}} < r \in \mathbb{Q}^+\}$ *is* Σ^0_N.

Let L_0 be the language that consists of countably many n-ary predicate symbols for each n.

Theorem (CGM 2021+)

There is a computably presentable L_0-structure $\mathcal{M}_0$ so that:

1. $\{(\phi, r) : \phi \text{ q.f. } \wedge\ \phi^{\mathcal{M}} < r \in \mathbb{Q}^+\}$ *is* Σ^0_1*-complete.*
2. $\{(\phi, r) : \phi \text{ q.f. } \wedge\ \phi^{\mathcal{M}} \leq r \in \mathbb{Q}^+\}$ *is* Π^0_1*-complete.*
3. *For every positive integer N:*
   1. $\{(\phi, r) : \phi \in \Pi_N \wedge\ \phi^{\mathcal{M}} \leq r \in \mathbb{Q}^+\}$ *is* Π^0_N*-complete.*
   2. $\{(\phi, r) : \phi \in \Pi_N \wedge\ \phi^{\mathcal{M}} < r \in \mathbb{Q}^+\}$ *is* Σ^0_{N+1}*-complete.*
   3. $\{(\phi, r) : \phi \in \Sigma_N \wedge\ \phi^{\mathcal{M}} \leq r \in \mathbb{Q}^+\}$ *is* Σ^0_{N+1}*-complete.*
   4. $\{(\phi, r) : \phi \in \Sigma_N \wedge\ \phi^{\mathcal{M}} < r \in \mathbb{Q}^+\}$ *is* Σ^0_N*-complete*

Definition

Let $\mathcal{M}$ be an L-structure.

1. $\mathrm{Th}(\mathcal{M}) : \mathrm{Sent}_L \to [0,1]$ defined by $\mathrm{Th}(\mathcal{M})(\phi) = \phi^{\mathcal{M}}$.
2. $\mathrm{Th}_{\forall_n}(\mathcal{M})$ is restriction of $\mathrm{Th}(\mathcal{M})$ to the Π_n sentences of L.

Corollary

If $\mathcal{M}$ is computably presentable, then:

1. $\emptyset^{(\omega)}$ *computes* $\mathrm{Th}(\mathcal{M})$.
2. *For each positive n,* $\emptyset^{(n)}$ *computes* $\mathrm{Th}_{\forall_n}(\mathcal{M})$.

Corollary (Goldbring-Hart, 2020)

If $\mathcal{M} \in \{C(2^\omega), C(\mathbb{P})\}$, *then :*

1. $\emptyset^{(\omega)}$ *computes* $\mathrm{Th}(\mathcal{M})$.
2. *For each positive integer n,* $\emptyset^{(n)}$ *computes* $\mathrm{Th}_{\forall_n}(\mathcal{M})$.

Observation

1. $\emptyset^{(\omega)}$ is the Turing degree of $\mathrm{Th}(\mathcal{M}_0)$. (That is, if X computes $\mathrm{Th}(\mathcal{M}_0)$, then X computes $\emptyset^{(\omega)}$.)
2. For each positive integer n, $\emptyset^{(n)}$ is the Turing degree of $\mathrm{Th}_{\forall_n}(\mathcal{M}_0)$.

Let $\mathcal{R}$ denote the hyperfinite II_1 factor.

Theorem (Goldbring, Hart 2020+)

The Turing degree of $\mathrm{Th}_{\forall_1}(\mathcal{R})$ *is* $\emptyset'$.

Proof use technology from proof of $\mathrm{MIP}^* = \mathrm{RE}$ result.

Questions

1. If $\mathcal{M}$ is a computably presentable L-structure, does $\mathrm{Th}_{\forall_n}(\mathcal{M})$ have a Turing degree?
2. For $n \geq 2$, does $\mathrm{Th}_{\forall_n}(\mathcal{R})$ have a Turing degree and if so is it $\emptyset^{(n)}$?
3. For which X is $C(X)$ computably presentable?