Interactions between computability and complex analysis

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Goals

- Computability in $\mathbb{C}$
- H^p
- Functions in H^p avoid two problems:
  - inability to compute f from its Taylor series
  - incomputable functions that are computable vectors in Banach spaces of analytic functions
- If time: Blaschke products

Computability in $\mathbb{C}$- points and sequences

Definition

$z \in \mathbb{C}$ is *computable* if there is an algorithm that given $k \in \mathbb{N}$ as input computes rational point $q \in \mathbb{C}$ so that $|q - z| < 2^{-k}$

Definition

$\{z_n\}_{n=0}^{\infty}$ is *computable* if there is an algorithm that given $n, k \in \mathbb{N}$ computes a rational point $q \in \mathbb{C}$ so that $|q - z_n| < 2^{-k}$.

Computability in $\mathbb{C}$- functions

Philosophy:

- Since $\mathbb{C}$ is uncountable, we can not compute with points in $\mathbb{C}$, we can only compute with approximations of points.
- To compute $f :\subseteq \mathbb{C} \to \mathbb{C}$ means that we should be able to compute arbitrarily good approximations of $f(z)$ from sufficiently good approximations of z.
- We use open rational rectangles as approximations of points.

Computability in $\mathbb{C}$- functions

Definition

$f :\subseteq \mathbb{C} \to \mathbb{C}$ is *computable* if there is an algorithm P with the following properties:

- **(Approximation)** Given an open rational rectangle R_1 as input, P either does not halt or produces an open rational rectangle R_2 as output.
- **(Correctness)** If $z \in R_1$, and if P halts on R_1 with output R_2, then $f(z) \in R_2$ whenever $z \in \mathrm{dom}(f) \cap R_1$.
- **(Convergence)** If U is a neighborhood of $f(z)$, then there is a rational rectangle R_1 so that $z \in R_1$ and so that on input R_1 P produces a rational rectangle included in U.

Examples: polynomials, sin, cos, exp, ...

n.b. $f :\subseteq \mathbb{C} \to \mathbb{C}$ computable $\Rightarrow$ f continuous.

Some basic complex analysis

Suppose: $f :\subseteq \mathbb{C} \to \mathbb{C}$, $\mathrm{dom}(f)$ open and connected.

- *f holomorphic* if $f'(z_0)$ exists at every $z_0 \in \mathrm{dom}(f)$.
- *f analytic* if for every $z_0 \in \mathrm{dom}(f)$ there exist $\{a_n\}_n$, $R > 0$ so that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)$$

when $|z - z_0| < R$

Fact: holomorphic $\Leftrightarrow$ analytic

Computably analytic functions

Definition

An open $U \subseteq \mathbb{C}$ is *computably open* if the set of all rational rectangles R so that $\overline{R} \subseteq U$ is computably enumerable.

Definition

f is *computably analytic* if f is analytic and computable and $\mathrm{dom}(f)$ computably open.

Theorem (Folklore)

If f is computably analytic then f' is computable.

Proof is 'ultra-uniform'.

Theorem (J. Myhill, 1971)

There is a computable function $f \in C^1[0,1]$ whose derivative is not computable.

Maclaurin Series

Definition

Suppose f analytic and $0 \in \text{dom}(f)$. Let

$$a_n(f) = \frac{f^{(n)}(0)}{n!}$$

So, for some $R > 0$

$$f(z) = \sum_{n=0}^{\infty} a_n(f) z^n$$

whenever $|z| < R$.

Maclaurin Series

Corollary

If f computably analytic then $\{a_n(f)\}_{n=0}^{\infty}$ computable.

Converse?

Notation

Let $\mathbb{D} = \{z : |z| < 1\}$ = unit disk.

Theorem (Caldwell & Pour-El 1975)

There is a function f that is analytic on $\mathbb{D}$ and incomputable even though $\{a_n(f)\}_{n=0}^{\infty}$ is computable.

Proof sketch: Let C be an incomputable *c.e.* set. Let $\{c_n\}_n$ be a 1-1 enumeration of C. Set

$$f(z) = \sum_{n=0}^{\infty} (1 + 2^{-c_n})^n z^n$$

$f(z)$ defined if $|z| < 1$ (use Ratio Test). To decide $n \in C$ compute $f(1/(1 + 2^{-n}))$.

Hardy spaces

Definition

Suppose $1 \leq p < \infty$. When f is analytic on $\mathbb{D}$, let

$$\begin{aligned} M_p(f;r) &= \text{the average of } |f|^p \text{ on } C_r(0) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta. \end{aligned}$$

Fact: $M_p(f;r)$ is non-decreasing as a function of r.

Definition

1. When $1 \leq p < \infty$, H^p consists of all analytic functions on $\mathbb{D}$ so that $M_p(f;r)$ bounded.
2. H^∞ consists of all bounded analytic functions on $\mathbb{D}$.

Fact: $p_1 < p_2 \Rightarrow \ H^{p_2} \subset H^{p_1}$.

H^p as a Banach space

Notation

1. *When* $1 \leq p < \infty$ *and* $f \in H^p$, *set*

$$\|f\|_{H^p} = \left(\sup_{0<r<1} M_p(f; r) \right)^{1/p}$$

2. *When* $f \in H^\infty$, $\|f\|_{H^\infty} = \sup\{|f(z)| \ : \ |z| < 1\}$.

Fact: If $1 \leq p \leq \infty$, H^p is a Banach space.

Theorem (M. 2014)

If $f \in H^p$, and if $\{a_n(f)\}_n$ is computable, then f is computable.

Proof sketch: We use following fact:

$$f \in H^1 \Rightarrow \lim_{n\to\infty} a_n(f) = 0.$$

So $\{|a_n(f)|\}_n$ bounded above by some $M > 0$.

$$\left|\sum_{n=k}^{\infty} a_n(f) z^n\right| \leq \sum_{n=k}^{\infty} M|z|^n$$

Since $|z| < 1$ can estimate errors by tails of geometric series.

Some computable Banach space theory

Intuitive idea: A *computable Banach space* is a Banach space $\mathcal{B} = (V, \cdot, +, \| \|)$ together with a notion of computability on V under which $\cdot$, $+$, $\| \|$ are computable.

Suppose $\mathcal{B} = (V, \cdot, +, \| \|)$ is a Banach space.

- $E \subseteq V$ is a *generating set* for $\mathcal{B}$ if $V = \overline{\mathrm{Span}(E)}$.
- If $v_1, \ldots, v_n \in V$ and $\alpha_1, \ldots, \alpha_n$ are rational points in $\mathbb{C}$, $\alpha_1 v_1 + \ldots + \alpha_n v_n$ is a *rational linear combination*.
- E is an *effective generating set* if it is a generating set and if it has an enumeration $\{e_n\}_{n=0}^{\infty}$ so that

$$\alpha_0, \ldots, \alpha_n \in \mathbb{Q}^2 \mapsto \| \sum_j \alpha_j e_j \|$$

is computable. (Call this a *suitable enumeration*.)

The 'typical' example $L^p[-\pi, \pi]$

Suppose $1 \leq p < \infty$

- $L^p[-\pi, \pi] =$ set of all measurable $f : [-\pi, \pi] \to \mathbb{C}$ so that $\int_{-\pi}^{\pi} |f(\theta)|^p d\theta < \infty$.
- $\|f\|_{L^p} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^p \right)^{1/p}$
- $e_n(\theta) = e^{in\theta}$ $n \in \mathbb{Z}$
- $E = \{e_n \ : \ n \in \mathbb{Z}\}$ is a generating set for $L^p[-\pi, \pi]$.
- When p is computable, E is an effective generating set since integration is a computable operator.

Turning effective generating sets into computability notion on $\mathcal{B}$

Fix an effective generating set for $\mathcal{B}$ and fix a suitable enumeration of E, $\{e_n\}$. **Intuition:** Parallel development of computability on $\mathbb{C}$

- Replace rational points with rational linear combinations
- Replace rational rectangles with *rational balls*:

$$B_r(\sum_{j=0}^{n} \alpha_j e_j)$$

$r \in \mathbb{Q}^+$ and $\alpha_j \in \mathbb{Q}^2$

Again define computability for points, sequences, functions

Definition

$v \in \mathcal{B}$ is *computable* if from $k \in \mathbb{N}$ one can compute rational linear combination $u = \sum_{j=0}^{n} \alpha_j e_j$ so that $\|u - v\| < 2^{-k}$.

Definition

$\{v_n\}_n$ *computable* if v_n computable uniformly in n.

Now, suppose we have computable Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ (possibly $=$)

Definition

$f : \mathcal{B}_1 \to \mathcal{B}_2$ *computable* if there is an algorithm P so that:

- **(Approximation)** Given a rational ball B_1 as input, P either does not halt or produces a rational ball B_2.
- **(Correctness)** $f[B_1] \subseteq B_2$
- **(Convergence)** If U is a neighborhood of $f(v)$, then there is a rational ball B_1 so that $v \in B_1$ and so that on input B_1, P produces a rational ball $B_2 \subseteq U$.

Therefore $\cdot$, $+$, $\| \ \|$ computable.

Computable vectors vs. computable functions

We want to work with Banach spaces of functions.

Problem: When working with Banach spaces of functions, computable *vectors* may not be computable *functions*.

e.g. $L^p[-\pi, \pi]$ contains discontinuous functions that are computable as vectors in L^p.

Theorem (Pour-El and Richards)

If $1 \leq p < \infty$, and if p computable, then there is a continuous incomputable function $f \in L^p[-\pi, \pi]$ so that f is computable as a vector in $L^p[-\pi, \pi]$.

Also note that:

Proposition ('trivial')

Suppose p is a computable real so that $1 \leq p < \infty$. If f is a computable function in $L^p[-\pi, \pi]$, then $\|f\|$ computable.

H^p as a computable Banach space

Assume $1 \leq p < \infty$.

- **Fact:** Polynomials are dense in H^p.
- This gives a generating set E: set $e_n(z) = z^n$.
- If p computable, this is an effective generating set:
  - Since $M_p(f; r)$ non-decreasing, can compute integral on boundary of unit disk when f a polynomial.
  - Again, use fact that integration is a computable operator.

Theorem (M. 2014)

If $1 \leq p < \infty$ is computable, and if $f \in H^p$ is computable as a vector in H^p, then f is computable (as a function).

To prove this theorem, it suffices to show the following:

Lemma

If $1 \leq p < \infty$ is computable, and if $f \in H^p$ is computable as a vector in H^p, then $\{a_n(f)\}_n$ is computable.

We can prove this now, but we want to make connections with some other ideas...

Radial limit functions

Fact: If $1 \leq p \leq \infty$, and if $f \in H^p$, then $\lim_{r \to 1^-} f(re^{i\theta})$ exists almost everywhere.

Notation

When $f \in H^p$, let $\tilde{f}(\theta) = \lim_{r \to 1^-} f(re^{i\theta})$.

Facts:

- $f \in H^p \Rightarrow \tilde{f} \in L^p[-\pi, \pi]$
- $f \mapsto \tilde{f}$ is a linear isometry of H^p with a subspace of $L^p[-\pi, \pi]$.
- Can recover f from $\tilde{f}$ via Poisson integral

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \tilde{f}(\theta)\, d\theta$$

Fourier coefficients

Notation

When $g \in L^p[-\pi, \pi]$, *let*

$$c_n(g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) e^{-in\theta} d\theta \; n \in \mathbb{Z}$$

Fact: If $1 \leq p < \infty$ and $f \in H^p$

$$c_n(\tilde{f}) = \begin{cases} a_n(f) & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

Proof of Lemma

Suppose $f \in H^p$ computable as a vector in H^p. Suffices to show $\{c_n(\tilde{f})\}_n$ computable.

Algorithm: Given n, k compute polynomial Q so that $\|f - Q\| < 2^{-k}$. Output $a_n(Q)$.

Verification: NTS $|c_n(\tilde{f}) - a_n(Q)| < 2^{-k}$. By direct calculation

$$|c_n(\tilde{f}) - c_n(\tilde{Q})| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{f}(\theta) - Q(e^{i\theta})| d\theta.$$

Since $p \geq 1$, by Jensen's inequality

$$\begin{aligned}\left(\int_{-\pi}^{\pi} |\tilde{f}(\theta) - Q(e^{i\theta})| \frac{d\theta}{2\pi}\right)^p &\leq \int_{-\pi}^{\pi} |\tilde{f}(\theta) - Q(e^{i\theta})|^p \, \frac{d\theta}{2\pi} \\ &= \|\tilde{f} - \tilde{Q}\|^p < (2^{-k})^p\end{aligned}$$

So $|c_n(\tilde{f}) - c_n(\tilde{Q})| < 2^{-k}$.

Theorem (M. 2014)

Suppose p is a computable real so that $p \geq 1$. Then, there is a computable function in H^p whose H^p-norm is incomputable.

Proof sketch: First consider the case $p = 2$.

$$\|f\|_{H^2} = \left(\sum_{n=0}^{\infty} |a_n(f)|^2 \right)^{1/2}.$$

So, choose computable sequence reals $\{b_n\}$ so that $\sum b_n^2$ an incomputable real. Set $f(z) = \sum_{n=0}^{\infty} b_n z^n$. By previous theorem, f computable.

Now suppose $p \neq 2$. Let C be a c.e. incomputable set with 1-1 enumeration $\{c_n\}_{n=0}^{\infty}$. Set $b_n = 2^{-c_n}$. Thus, $\sum_{n=0}^{\infty}(b_n)^k$ is incomputable for all $k \in \mathbb{N}$. Set

$$g_1(z) = \sum_{n=0}^{\infty} b_n z^n$$

$g_1 \in H^\infty$ so g is computable. Now choose rational number $M > \sup\{|g_1(z)| \ : \ |z| < 1\}$. Set $g = M + g_1$. So, $g \in H^\infty$, g computable, g has no zeros. Compute an analytic branch of $g^{2/p}$ and set $f = g^{2/p}$.

Note $\|g\|_{H^2}$ incomputable. However

$$(\|f\|_{H^p})^p = (\|g\|_{H^2})^2$$

So, $\|f\|_{H^p}$ incomputable.

Blaschke products

- When $a \neq 0$, $b_a(z) = \frac{|a|}{a} \frac{a-z}{1-\bar{a}z}$.
- When $A = \{a_n\}_{n=0}^{\infty}$ is a sequence of non-zero complex numbers, let $B_A = \prod_{n=0}^{\infty} b_{a_n}$.
- $B_{A,m}(z) = z^m B_A(z)$
- Let $\Sigma_A = \sum_{n=0}^{\infty} 1 - |a_n|$.
- Blaschke sequence $\Sigma_A < \infty$.

Theorem (Matheson, M. 2008)

There is a computable Blaschke sequence A such that B_A is not computable. (In particular, $B_A(0)$ is an incomputable real.)

Theorem (E. Specker, 1959)

There is a computable, decreasing, and convergent sequence of positive rational numbers $1 > r_0 > r_1 > r_2 > \ldots > 0$ *whose limit is incomputable.*

Set

$$\begin{aligned} a_0 &= r_0 \\ a_{n+1} &= \frac{r_{n+1}}{r_n} \end{aligned}$$

Thus,

$$B_A(0) = \prod_{n=0}^{\infty} a_n = \lim_{n\to\infty} r_n.$$

Thus, $\sum_{n=0}^{\infty} 1 - |a_n| < \infty$.

Theorem (Matheson, M. 2008)

If $z_0 \in (0,1)$ is computable, then there is a computable Blaschke sequence A such that $B_A(z_0)$ is not computable.

Proposition (Andreev, M. 2010)

There is a computable and interpolating sequence A such that B_A is not computable.

Question

If B is a computable Blaschke product, is there a computable Blaschke sequence A such that $B_A = B$?

Theorem (Matheson, M. 2008)

If B is a computable Blaschke product, then there is a computable A such that $B = B_A$.

Theorem (M. 2008)

If A, Σ_A are computable, then B_A is computable.

Proof ‘ultra uniform’

Theorem (M. 2008)

If A and $B_A(0)$ are computable, then Σ_A is computable.


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