“How to hide from a nanobot”

Timothy H. McNicholl

Department of Mathematics
Lamar University
timothy.h.mcnicholl@gmail.com
http://sites.google.com/site/timothymcnicholl

January, 2012

Two graph theory problems

- Traveling Salesman Problem
- Hamiltonian Circuit Problem

Continuous versions

Question (Analyst's Traveling Salesman Problem)

Which closed subsets of $\mathbb{R}^n$ are contained in a rectifiable curve?

- Answered for $n = 2$ by P. Jones in 1990.
- Answered for all n by K. Okikioulu in 1992.

Question (Continuous Hamiltonian Circuit)

Which closed subsets of $\mathbb{R}^n$ are contained in an arc?

(arc $=$ homeomorphic image of $[0, 1]$)

- Solved for $n = 2$ by R.L.Moore and J.R. Kline in 1919

Question (Computable Analyst's Traveling Salesman Problem)

Which points in $\mathbb{R}^n$ belong to a computable rectifiable curve?

(Curve if computable if it is range of a computable $f : [0, 1] \to \mathbb{R}^n$.)

- Solved by Gu, Lutz, Mayordomo 2006

Question (Computable Analyst's Hamiltonian Circuit Problem)

Which points in $\mathbb{R}^n$ lie on a computable arc?

(Arc is computable if it is range of computable and injective $f : [0, 1] \to \mathbb{R}^n$.)

- Not solved yet, but lots of interesting partial results.

Question

How are these classes related?

Theorem (McNicholl (and Lutz) '11)

There is a point on a computable arc in $\mathbb{R}^2$ that does not lie on any computable rectifiable curve.

Proof.

First proof (due to J. Lutz)

- If a point lies on a computable rectifiable curve, then its effective Hausdorff dimension is at most 1.
- There is a computable arc A whose Hausdorff dimension is > 1. (Lots of them!)
- Therefore there is a point on A whose effective Hausdorff dimension is > 1. (Lutz, 2005)

$\square$

Proof.

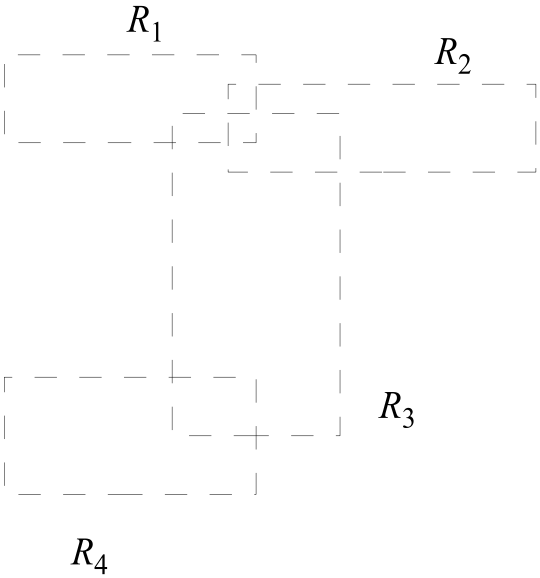
Second proof:

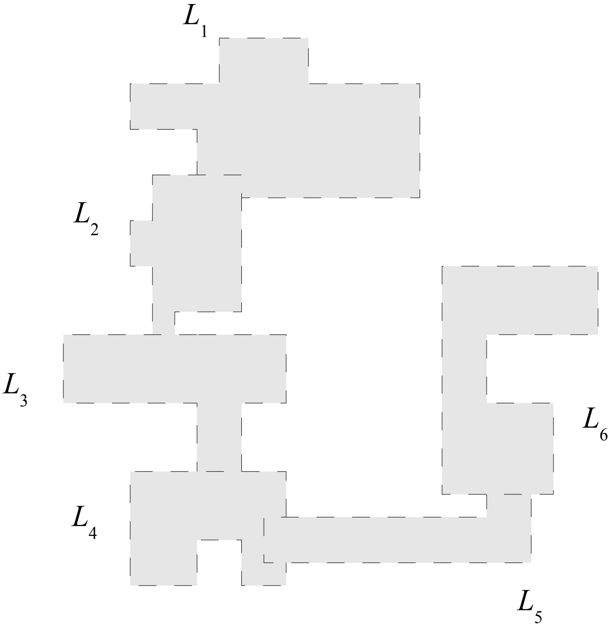
- For every $k \in \mathbb{N}$, there is a computable arc $A_k \subseteq [0,1]^2$ whose measure is $\geq 1 - 2^k$. (Osgood 19??).
- A_k can be computed uniformly from k.
- Therefore, $[0,1]^2 - \bigcup_k A_k$ has effective measure 0.
- So, every Martin-Löf random point belongs to an A_k.
- No Martin-Löf random point belongs to a computable rectifiable curve (Gu, Lutz, Mayordomo 2006).

$\square$

Theorem (McNicholl '11)

There is a point on a computable and computably rectifiable curve that does not belong to any computable arc.


R_1
R_2
R_3
R_4


L_1
L_2
L_3
L_4
L_5
L_6

Definition

An arc chain $\mathcal{C}_1$ *refines* an arc chain $\mathcal{C}_2$ if each link of $\mathcal{C}_1$ is contained in a link of $\mathcal{C}_2$ and if each link of $\mathcal{C}_2$ has the property that the links of $\mathcal{C}_1$ it contains form an arc chain.

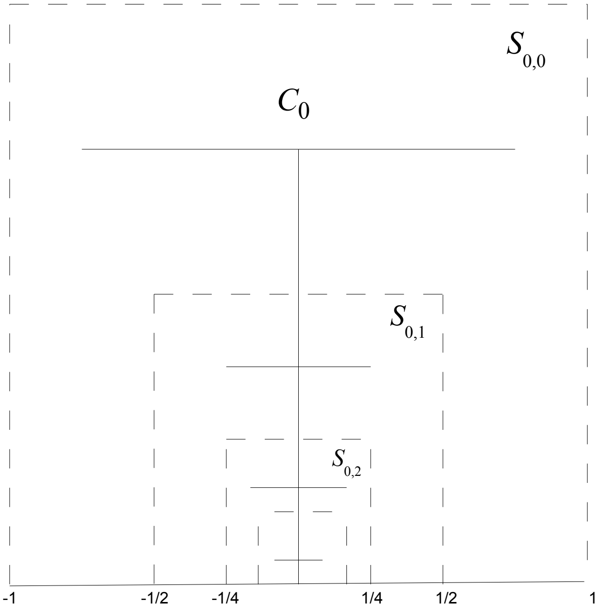
Theorem (sort of in Daniel & McNicholl '11 and in McNicholl '11)

An arc A is computable (i.e. has a computable parameterization) if and only if there is a computable sequence of arc chains $\mathcal{C}_1, \mathcal{C}_2, \ldots$ such that:

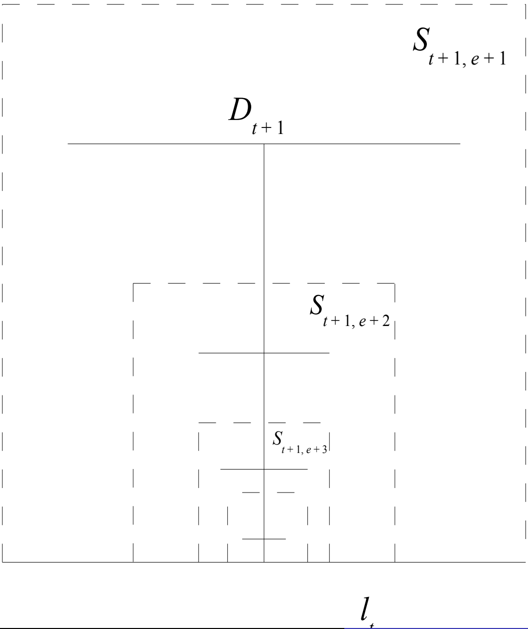
1. $\mathcal{C}_{j+1}$ *refines* $\mathcal{C}_j$,
2. *the diameter of each link of* $\mathcal{C}_j$ *is smaller than* 2^{-j}, *and*
3. *A is contained in the union of the links of* $\mathcal{C}_j$.

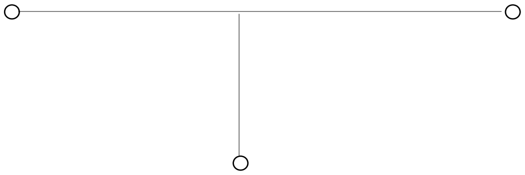
Proof is uniform and relativizes.

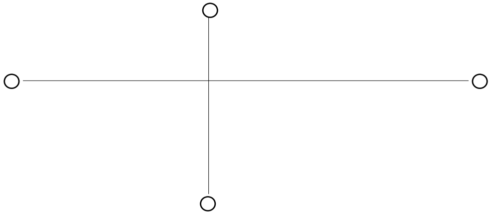
Proof sketch: Use a finite injury argument.



Stage $t+1$:







$p := \lim_{t \to \infty} p_t$ does not lie on any computable arc but lies on $C := \bigcup_t C_t$ which has computable length.

Definition

A set $X \subseteq \mathbb{R}^2$ is *effectively locally connected* if there is a Turing machine with a one-way output tape and two input tapes and with the property that whenever a name of a point $p \in X$ is written on the first input tape and a name of an open set $U \subseteq \mathbb{R}^2$ that contains p is written on the second input tape and the machine is allowed to run indefinitely, a name of an open set $V \subseteq \mathbb{R}^2$ for which $p \in V$ and $V \cap X$ is connected is written on the output tape.

Lemma

C is computably compact and ELC.

Theorem (Couch, Daniel, and McNicholl '10)

If $X \subseteq \mathbb{R}^n$ is computably compact and ELC then there is a computable surjection of $[0, 1]$ onto X.

Theorem (Couch, Daniel, McNicholl '10)

There is a computable map $f : [0, 1] \to \mathbb{R}^2$ whose range is not ELC.

- P.J. Couch, B.D. Daniel, and T.H. McNicholl, *Computing space-filling curves*, To appear in Theory of Computing Systems. Available online at www.springerlink.com. DOI: 10.1007/s00224-010-9306-3.
- D. Daniel and T.H. McNicholl, *Effective local connectivity properties*, To appear in Theory of Computing Systems. DOI:10.1007/s00224-011-9364-1.
- X. Gu, J. Lutz, and E. Mayordomo, *Points on computable curves*, Proceedings of the Forty-Seventh Annual IEEE Symposium on Foundations of Computer Science (Berkeley, CA), IEEE Computer Society Press, October 2006, pp. 469–474.
- Peter W. Jones, *Rectifiable sets and the traveling salesman problem*, Invent. Math. **102** (1990), no. 1, 1–15.
- Jack H. Lutz, *Effective fractal dimensions*, MLQ Math. Log. Q. **51** (2005), no. 1, 62–72.

T.H. McNicholl, *The power of backtracking and the confinement of length*, To appear in Proceedings of the American Mathematical Society.

R.L. Moore and J.R. Kline, *On the most general plane closed point-set through which it is possible to pass a simple continuous arc*, Ann. of Math. **20** (1919), no. 3, 218–223.

Kate Okikiolu, *Characterization of subsets of rectifiable curves in* $\mathbf{R}^n$, J. London Math. Soc. (2) **46** (1992), no. 2, 336–348.

Hans Sagan, *Space-filling curves*, Universitext, Springer-Verlag, New York, 1994.