An application of computable probability to computable analysis

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- Recall that a *Banach space* is a normed vector space that is complete in the metric induced by the norm.
- The field of scalars of a Banach space can be either $\mathbb{R}$ or $\mathbb{C}$. In this talk, we only consider real Banach spaces BUT all results hold for complex Banach spaces as well.

Example

$\mathbb{R}^n$ is a Banach space under the Euclidean norm $\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \ldots + x_n^2}$.

Example

Let $C[0,1]$ denote the set of all continuous real-valued functions on $[0,1]$. $C[0,1]$ is a vector space under pointwise addition and scalar multiplication. (i.e. $(f+g)(t) = f(t) + g(t)$ and $(\alpha f)(t) = \alpha f(t)$.) It is a Banach space under the supremum norm (i.e. $\|f\| = \max\{|f(t)| \ : \ t \in [0,1]\}$).

Review of Lebesgue spaces

Motivation

Recall from calculus $\int_0^1 \frac{1}{x^p} dx < \infty$ *if and only if* $p < 1$. L^p *spaces are the ultimate generalization of this sort of problem.*

Definition

Let $\Omega = (X, \mathcal{S}, \mu)$ be a measure space, and suppose $1 \leq p < \infty$. Then, $L^p(\Omega) =$ set of all measurable $f : X \to \mathbb{F}$ so that $\int |f|^p \, d\mu < \infty$.

Remark

Assume $1 \leq p < \infty$.

1. $L^p(\Omega)$ *is a vector space under pointwise addition and scalar multiplication.*
2. *We identify functions in* $L^p(\Omega)$ *that agree a.e..*
3. *If* $p < \infty$, *then* $L^p(\Omega)$ *is a Banach space under the norm*

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{1/p}.$$

Review of Lebesgue spaces

Definition

1. An L^p *space* is a Banach space of the form $L^p(\Omega)$.
2. A Banach space is a *Lebesgue* space if it is an L^p space for some $p \geq 1$.

Example

1. $L^p[0,1] = L^p([0,1], \mathcal{S}, m)$ where $\mathcal{S}$ is the σ-algebra of Lebesgue measurable subsets of $[0,1]$ and $m =$ Lebesgue measure on $[0,1]$.
2. $\ell^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where μ is the counting measure on $\mathbb{N}$. (so $\left\|\{s_n\}_{n=0}^{\infty}\right\|_p = \left(\sum_{n=0}^{\infty} |s_n|^p\right)^{1/p}$).
3. $\ell_n^p = L^p(\{1, \ldots, n\}, \mathcal{P}(\mathbb{N}), \mu)$ where μ is the counting measure on $\{1, \ldots, n\}$.

L^p spaces are common objects of study in various branches of analysis.

Definition

Let $\mathcal{B}$ be a Banach space.

1. A *presentation* of $\mathcal{B}$ is a pair $(\mathcal{B}, \{v_n\}_{n\in\mathbb{N}})$ where $\{v_n\}_{n\in\mathbb{N}}$ is a linearly dense sequence of vectors of $\mathcal{B}$ (i.e. $\mathcal{B}$ is the closed linear span of $\{v_0, v_1, \ldots\}$).
2. If $\mathcal{B}^{\#} = (\mathcal{B}, \{v_n\}_{n\in\mathbb{N}})$ is a presentation of $\mathcal{B}$, then we refer to v_n as the *n-th distinguished vector of* $\mathcal{B}^{\#}$.

Thus, to define a presentation of a Banach space, it suffices to specify the distinguished vectors.

Definition

Suppose $\mathcal{B}^\#$ is a presentation of a Banach space $\mathcal{B}$. A *rational vector* of $\mathcal{B}^\#$ is a rational linear combination of distinguished vectors of $\mathcal{B}^\#$; i.e., a vector that can be expressed in the form $\sum_{j=0}^{n} \alpha_j v_j$ where each α_j is a rational scalar and each v_j is a distinguished vector of $\mathcal{B}^\#$.

Definition

A presentation $\mathcal{B}^\#$ is *computable* if the norm function is computable on the rational vectors of $\mathcal{B}^\#$; i.e. there is an algorithm that given a $k \in \mathbb{N}$ and a (code of) a rational vector v of $\mathcal{B}^\#$, computes a rational number q so that $|q - \|v\|_{\mathcal{B}}| < 2^{-k}$. A code of such an algorithm is called an *index* of the presentation.

Remark

Among all presentations of a Banach space $\mathcal{B}$, one may be designated as standard; in this case, we will identify $\mathcal{B}$ with its standard presentation.

Example

1. The standard presentation of $\mathbb{R}$ is defined by declaring 1 to be the n-th distinguished vector for all n.
2. The standard presentations of ℓ^p_n and ℓ^p are defined via the standard bases.
3. The standard presentation of $L^p[0,1]$ is defined via the indicator functions of dyadic subintervals of $[0,1]$.
4. The standard presentation of $L^p(0,1)$ is defined similarly.
5. The standard presentation of $L^p((0,1)^\omega)$ is defined by declaring the $\langle n,k\rangle$-th distinguished vector to be the indicator function of $(0,1)^n \times I_k \times (0,1)^\omega$ where I_k is the k-th dyadic subinterval of $(0,1)$.

Note that these presentations are computable if p is.

Question

If there is a computably presentable L^p space whose dimension is at least 2, does it follow that p is computable?

Theorem (M. 2019+)

If $2 \leq p < \infty$, and if there is a computable presentation of an L^p space whose dimension is at least 2, then p is computable.

Theorem (M. 2019+)

If $n \geq 2$, and if ℓ_n^p is computably presentable where $1 \leq p < \infty$, then p is computable.

Theorem (Non-Uniformity Theorem (M. 2019+))

Suppose $I \subset [1,2]$ *is an open interval. There is no computable* $f :\subseteq \mathbb{N} \to \mathbb{N}$ *so that for all* $e \in \mathbb{N}$*, if* e *is the index of a presentation of a Lebesgue space* $\mathcal{B}$ *whose index lies in* I*, then* $f(e)$ *is an index of the exponent of* $\mathcal{B}$*.*

Definition

1. A sequence $(\mathcal{B}_e^{\#})_{e\in\mathbb{N}}$ of Banach space presentations is *uniformly computable* if there is a computable $g : \mathbb{N} \to \mathbb{N}$ so that $g(e)$ is an index of $\mathcal{B}_e$ for all e.
2. A sequence $(\mathcal{B}_e)_{e\in\mathbb{N}}$ of Banach spaces is *uniformly computably presentable* if there is a uniformly computable sequence $(\mathcal{B}_e^{\#})_{e\in\mathbb{N}}$ of presentations.

Proposition (M. 2019+)

Suppose $\mathcal{B}_0^{\#}$ and $\mathcal{B}_1^{\#}$ are computable presentations of Banach spaces. Let $A \subseteq \mathbb{N}$ be c.e., and for all $e \in \mathbb{N}$, let

$$\mathcal{A}_e = \begin{cases} \mathcal{B}_1 & e \notin A \\ \mathcal{B}_0 & e \in A \end{cases}$$

If there is a computable isometric embedding of $\mathcal{B}_1^{\#}$ into $\mathcal{B}_0^{\#}$, then $(\mathcal{A}_e)_{e \in \mathbb{N}}$ is uniformly computably presentable.

Theorem (Embedding Theorem (M. 2019+))

Suppose p, r are computable reals so that $1 \leq p < r \leq 2$. Then, there is a computable isometric embedding of ℓ^r into $L^p[0,1]$.

Definition

Let X be a random variable.

1. X is said to be *stable* if for all positive A, B, there is a positive C and a real D so that whenever X_1 and X_2 are independent random variables and $X_1 =_d X_2 =_d X$, $AX_1 + BX_2 =_d CX + D$.
2. If, in addition, we can always choose D to be zero, then X is *strictly stable*.

Theorem

Suppose X is stable. Then there is a unique real number $r \in (0, 2]$ so that whenever $A, B > 0$ and X_1, X_2 are independent copies of X (i.e. $X_1 =_d X_2 =_d X$),
$AX_1 + BX_2 =_d (A^r + B^r)^{1/r} X + D$ *for some $D \in \mathbb{R}$.*

When X is a random variable, let ϕ_X denote the characteristic function of X; that is $\phi_X(t) = E[e^{iXt}]$. Recall that two random variables are identically distributed if and only if they have the same characteristic function.

Theorem

Let X be a random variable.

1. *If X is r-stable, then there exist unique $\sigma \geq 0$, $\beta \in [-1, 1]$, and $\delta \in \mathbb{R}$ so that*

$$\phi_X(t) = \begin{cases} \exp\left(-\sigma^\alpha |t|^\alpha (1 - i\beta(\operatorname{sgn}(t)\tan(\frac{\pi\alpha}{2})) + i\delta t)\right) & \alpha \neq 1 \\ \exp\left(-\sigma|t|(1 + i\beta\frac{2}{\pi}\operatorname{sgn}(t)\ln|t|) + i\delta t\right) & \alpha = 1 \end{cases} \tag{1}$$

2. *If Φ_X has the form in Equation (1), then X is r-stable.*

The numbers σ, δ, and β are known as the *scale*, *shift*, and *skewness* parameters of X as well as its distribution.

Theorem (M. 2019)

If μ is a stable distribution, and if the parameters of μ are computable, then there is a computable random variable on $(0, 1)$ whose distribution is μ.

Theorem (M. 2019)

Let X be an r-stable random variable, and suppose the parameters of the distribution of X are computable.

1. *If $r < 2$, and if p is a computable real so that $0 \leq p < r$, then $E[|X|^p]$ is computable.*
2. *If $r = 2$, then $E[|X|^p]$ is computable for every nonnegative real p.*

Corollary (M. 2019+)

Suppose p, r are computable reals so that $1 \leq p < r \leq 2$. Let μ be a stable distribution whose parameters are computable and whose stability index is r. Then, $L^p((0,1);\mathbb{R})$ contains a computable vector whose distribution is μ.

Definition

Suppose $1 \leq p < \infty$, $\mathcal{B}$ is a Banach space, and $v_1, \ldots, v_n \in \mathcal{B}$. We say $v_1, \ldots, v_n$ are *L^p-formally disjointly supported* if

$$\left\| \sum_j \alpha_j v_j \right\|_{\mathcal{B}}^p = \sum_j |\alpha_j|^p \left\| v_j \right\|_{\mathcal{B}}^p$$

for all scalars $\alpha_1, \ldots, \alpha_n$. We say that a sequence $\{v_n\}_{n \in \mathbb{N}}$ of vectors is L^p-formally disjointly supported if $v_0, \ldots, v_M$ are L^p-formally disjointly supported for all M.

The choice of terminology in Definition 21 is motivated by the following two facts. First, if $f_1, \ldots, f_n \in L^p(\Omega)$ are disjointly supported, then they are L^p-formally disjointly supported. By a result of J. Lamperti, if $p \neq 2$, then L^p-formally disjointly supported vectors in $L^p(\Omega)$ are disjointly supported.

Lemma (M. 2019+)

Suppose $1 \leq p < r \leq 2$. *If* $\{g_n\}_{n\in\mathbb{N}}$ *is an independent family of symmetric and strictly* r*-stable random variables in* $L^p(\Omega;\mathbb{R})$, *then* $\{g_n\}_{n\in\mathbb{N}}$ *is* L^r*-formally disjointly supported.*

Lemma (M. 2019+)

Suppose $1 \leq p < r \leq 2$. *There is a computable sequence* $\{g_n\}_{n\in\mathbb{N}}$ *of* $L^p((0,1)^\omega;\mathbb{R})$ *so that each* g_n *is symmetric strictly* r*-stable and so that* $\{g_n\}_{n\in\mathbb{N}}$ *is independent.*

Proof of Embedding Theorem (sketch).

By above Lemmas, there is a computable sequence $\{g_n\}_{n\in\mathbb{N}}$ of $L^p((0,1)^\omega)$ that is L^r-formally disjointly supported and so that each g_n is nonzero. Let $h_n = \|g_n\|_p^{-1} g_n$.
For each $f \in \ell^r$, let $T_1(f) = \sum_n f(n)h_n$. By definition, T_1 is linear. Since $\{g_n\}_{n\in\mathbb{N}}$ is L^r-formally disjointly supported, T_1 is an isometry. By definition, T_1 maps the standard basis of ℓ^r onto a computable sequence of $L^p((0,1)^\omega)$. It follows that, T_1 is computable.
By a Theorem of Clanin, McNicholl, and Stull, there is a computable isometric isomorphism of $L^p((0,1)^\omega)$ onto $L^p[0,1]$.
Let $T = T_2 \circ T_1$. Thus, T is a computable isometric embedding of ℓ^r into $L^p[0,1]$. □

Proof of Non-Uniformity Theorem.

By way of contradiction, suppose such a function exists. Fix rational numbers $r_1, r_2 \in I$ so that $r_1 < r_2$. Let k_0 be the least natural number so that $r_2 - 2^{-k_0} > r_1$. Fix a computable surjection $\nu_{\mathbb{Q}}$ of $\mathbb{N}$ onto $\mathbb{Q}$. Let S denote the set of all $e \in \text{dom}(f)$ so that $k_0 \in \text{dom}(\phi_{f(e)})$ and $|\nu_{\mathbb{Q}}(\phi_{f(e)}(k_0)) - r_2| < 2^{-k_0}$. Thus, S is c.e..
Let

$$\mathcal{B}_e = \begin{cases} L^{r_1}[0,1] & e \in S \\ \ell^{r_2} & e \notin S \end{cases}$$

Thus, $(\mathcal{B}_e)_{e \in \mathbb{N}}$ is uniformly computably presentable. Thus, there is a computable $g : \mathbb{N} \to \mathbb{N}$ so that $g(e)$ indexes a presentation of $\mathcal{B}_e$ for all $e \in N$. By the Recursion Theorem, there is an $e_0 \in \mathbb{N}$ so that $\phi_{g(e_0)} = \phi_{e_0}$. Thus, e_0 indexes $\mathcal{B}_{e_0}$. Therefore, $f(e_0)$ is defined. Since $e_0 \in \text{dom}(f)$, if $e_0 \notin S$, then $f(e_0)$ does not index r_2. Thus, $\mathcal{B}_{e_0} = L^{r_1}[0,1]$, and $e_0 \in S$. But, then e_0 does not index r_1- a contradiction. □