Blaschke products and inner functions from the viewpoint of the theory of computation

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Computability theory/ theory of computation

Computability theory/ theory of computation

- Limitations of computers (*i.e.* discrete computing devices)
- When a problem can be solved by a computer, how difficult is it to solve? (in terms of time/memory) **Complexity Theory**
- When a problem can not be solved by any computer, how impossible is it (relative to other impossible problems)?

Fundamental notion: algorithm

- *i.e.* a procedure that can be carried out without thinking.
- Viewpoint: computers are just devices for implementing algorithms, so algorithms are the real focus of study.

Fundamental definition

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Definition

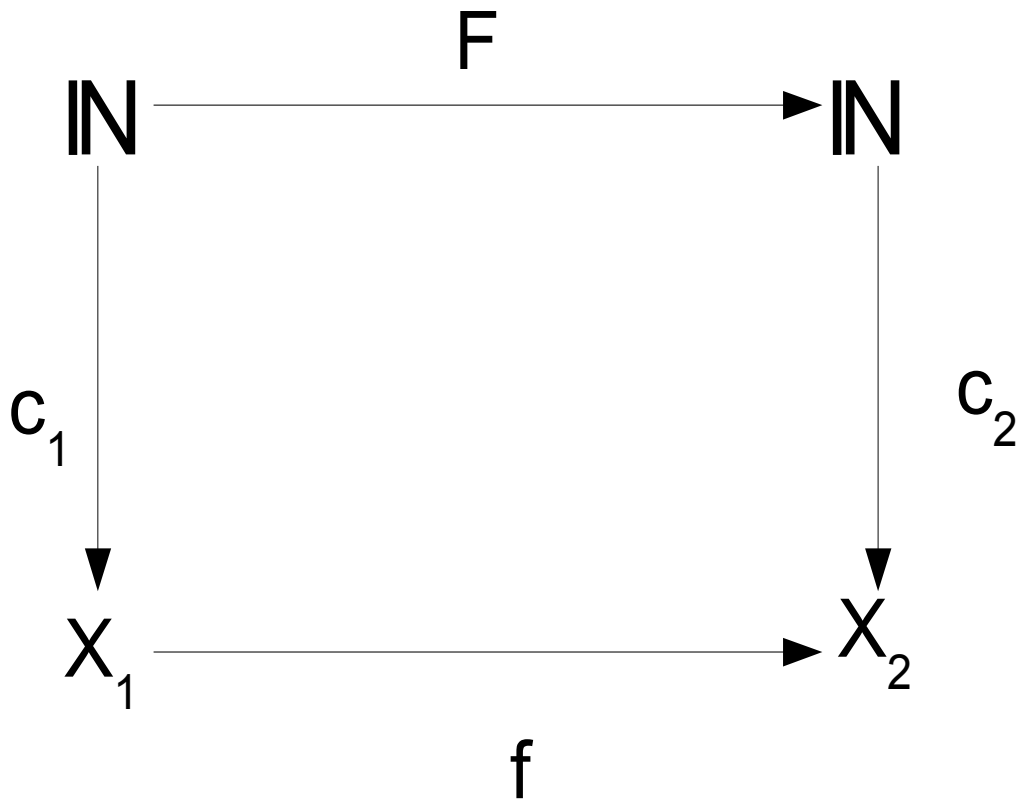
A function $f :\subseteq \mathbb{N}^m \to \mathbb{N}$ is *computable* if there is an algorithm that given any $n_1, \ldots, n_m \in \mathbb{N}$ as input, halts with output $f(n_1, \ldots, n_m)$ if $(n_1, \ldots, n_m) \in \mathrm{dom}(f)$ and does not halt if $(n_1, \ldots, n_m) \notin \mathrm{dom}(f)$.

Examples:

- Addition, multiplication, division
- gcd's
- probably any function you can think of.

Definition

A *coding* of a set X is a function $c :\subseteq \mathbb{N} \to X$ that is onto.



By means of codings, can extend computability to

- $\mathbb{Z}$
- $\mathbb{Q}$
- set of all finite graphs, *etc.*

ALAN TURING YEAR

2012

Formalizing ‘algorithm’

- Turing machines (A. Turing)
- Partial recursive functions (A. Church)
- Unlimited register machine (J. Von Neumann)
- Flowchart computability (Wang)

These are all equivalent!

Computably enumerable vs. computable

Definition

$A \subseteq \mathbb{N}$ is *computable* if there is an algorithm that given any $n \in \mathbb{N}$ as input determines if $n \in A$.

Definition

$A \subseteq \mathbb{N}$ is *computably enumerable* if either $A = \emptyset$ or if there is a sequence $\{a_n\}_{n \in \mathbb{N}}$ such that $A = \{a_0, a_1, a_2, \ldots\}$ and such that $n \mapsto a_n$ is a computable function.

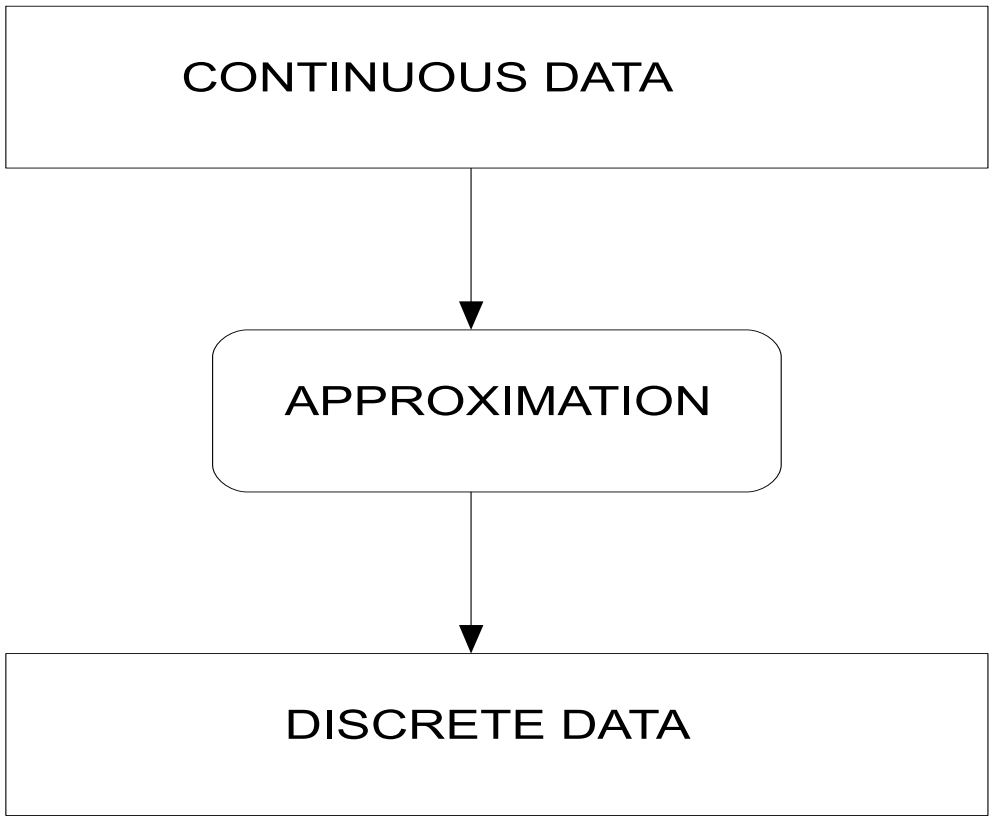
Theorem (Church, Turing, Kleene, Gödel, 1936)

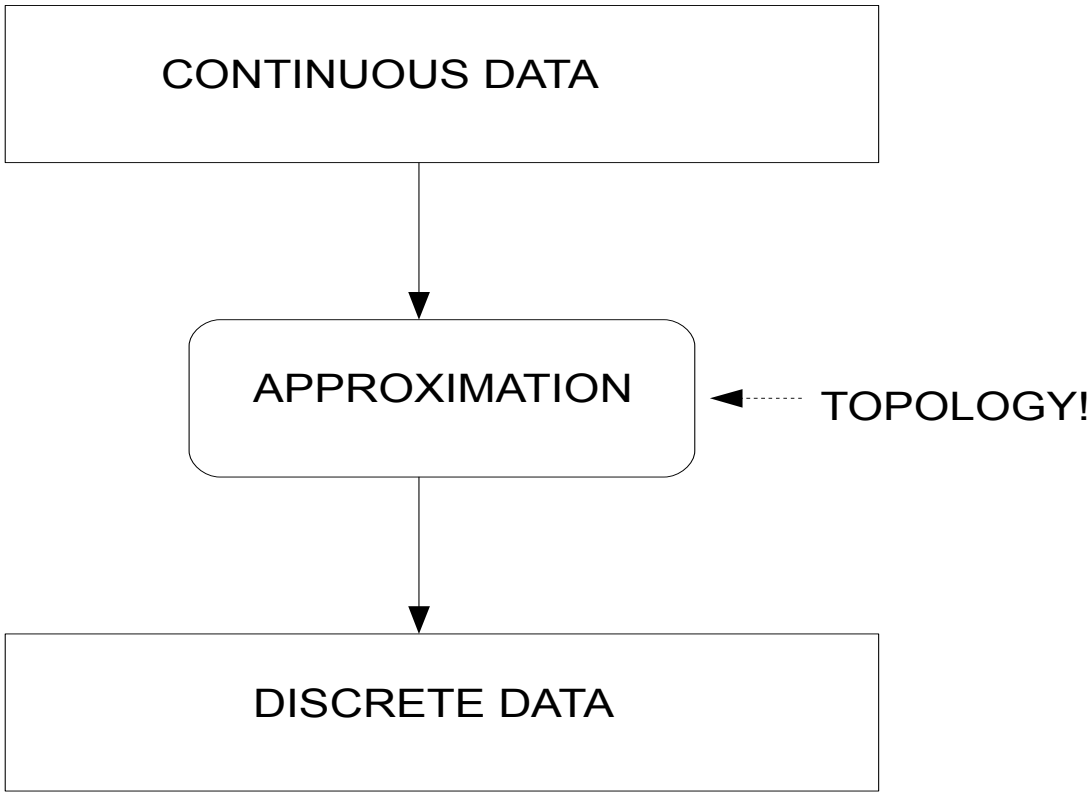
There is a computably enumerable set that is not computable.

Alec Matheson, 1946 - 2006

Computable analysis

Theory of computation with *continuous* data


CONTINUOUS DATA
APPROXIMATION
DISCRETE DATA


CONTINUOUS DATA
APPROXIMATION
TOPOLOGY!
DISCRETE DATA

Some notation

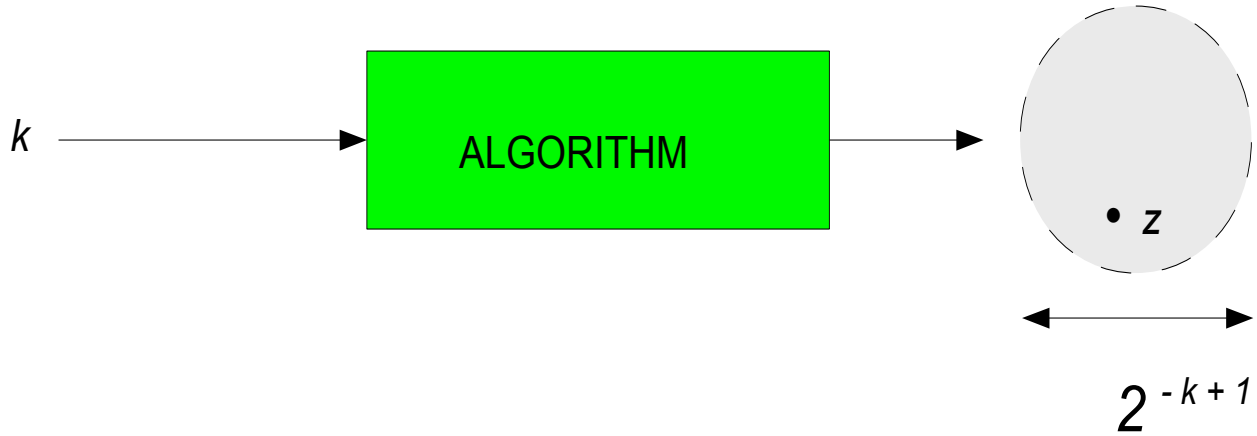
- $\mathbb{D} = \{z \ : \ |z| < 1\}$.
- When $a \neq 0$, $b_a(z) = \frac{|a|}{a} \frac{a-z}{1-\bar{a}z}$.
- When $A = \{a_n\}_{n=0}^{\infty}$ is a sequence of non-zero complex numbers, let $B_A = \prod_{n=0}^{\infty} b_{a_n}$.

Question

Is there a computable Blaschke sequence A such that B_A is not a computable function?

Definition

A point $z \in \mathbb{C}$ is *computable* if there is an algorithm that, given any $k \in \mathbb{N}$ as input, produces a rational point q such that $|z - q| < 2^{-k}$.



Examples:

- Any rational point.
- $e, \pi, \sqrt{2}$
- Almost any point you can think of.

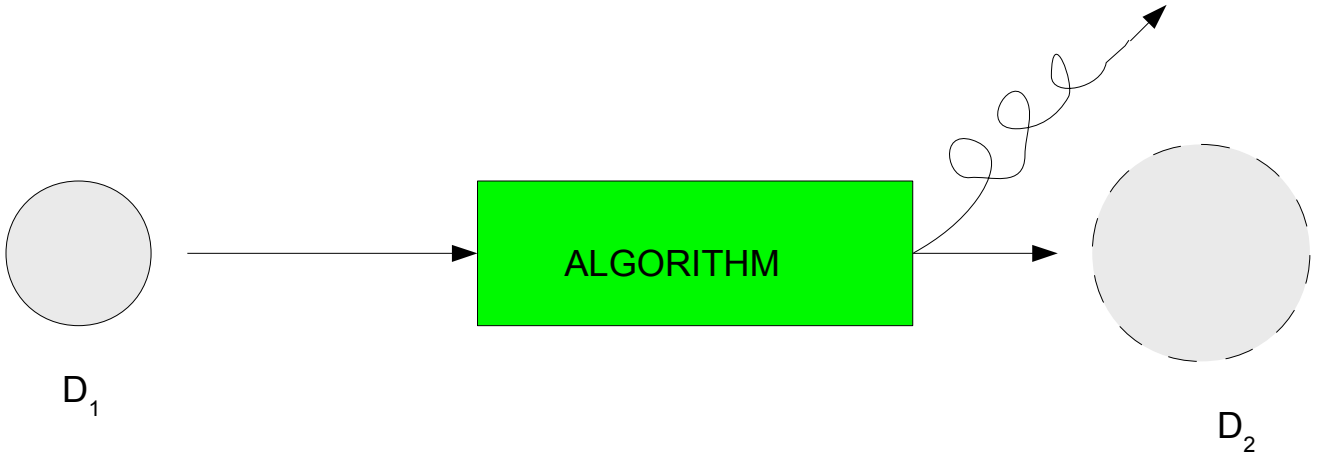
Definition

A sequence of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ is *computable* if there is an algorithm that, given any $k, n \in \mathbb{N}$ as input, produces a rational point q such that $|z_n - q| < 2^{-k}$.

Definition

Let $f : \mathbb{D} \to \mathbb{C}$. f is *computable* if there is an algorithm P that has the following three properties:

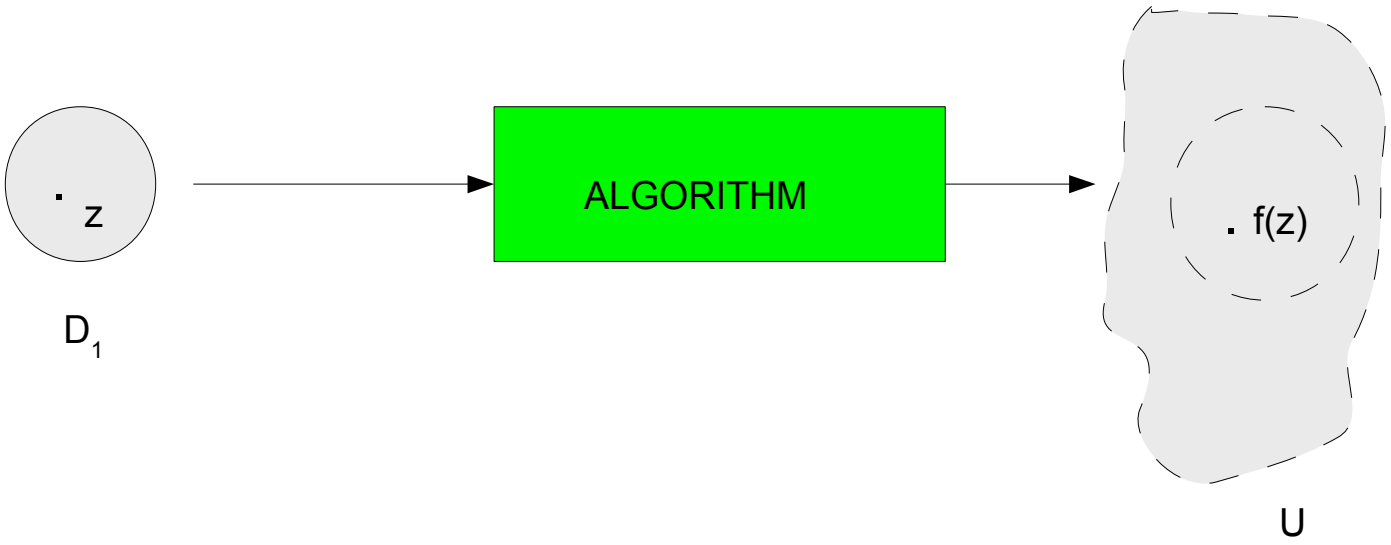
Property 1: approximation



Property 2: correctness

$$f[D_1] \subseteq D_2$$

Property 3: convergence



Proposition

If $f : \mathbb{D} \to \mathbb{C}$ is computable, and if $z \in \mathbb{D}$ is computable, then $f(z)$ is computable.

Theorem (Matheson, M. 2008)

There is a computable Blaschke sequence A such that B_A is not computable. (In particular, $B_A(0)$ is an incomputable real.)

Theorem (E. Specker, 1959)

There is a computable, decreasing, and convergent sequence of positive rational numbers $1 > r_0 > r_1 > r_2 > \ldots > 0$ *whose limit is incomputable.*

Set

$$\begin{aligned} a_0 &= r_0 \\ a_{n+1} &= \frac{r_{n+1}}{r_n} \end{aligned}$$

Thus,

$$B_A(0) = \prod_{n=0}^{\infty} a_n = \lim_{n\to\infty} r_n.$$

Thus, $\sum_{n=0}^{\infty} 1 - |a_n| < \infty$.

Theorem (Matheson, M. 2008)

If $z_0 \in (0,1)$ is computable, then there is a computable Blaschke sequence A such that $B_A(z_0)$ is not computable.

Proposition (Andreev, M. 2010)

There is a computable and interpolating sequence A such that B_A is not computable.

Question

If B is a computable Blaschke product, is there a computable Blaschke sequence A such that $B_A = B$?

Theorem (Matheson, M. 2008)

If B is a computable Blaschke product, then there is a computable A such that $B = B_A$.

So, suppose B is a computable Blaschke product. We need to compute a sequence A such that

- the terms of A are precisely the zeros of B,
- the number of times a term is repeated in A is its multiplicity as a zero of B.

Two key principles of computable analysis:

- Integration is a computable operator.
- Differentiation is not.

Theorem (J. Myhill, 1971)

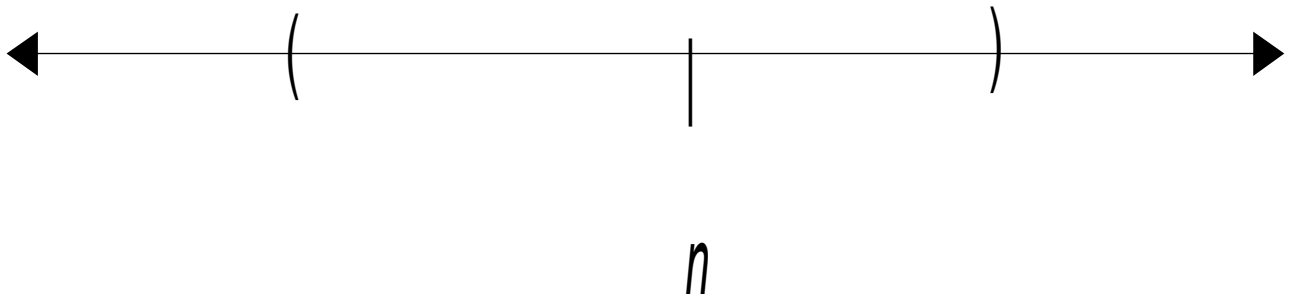
There is a computable function $f \in C^1[0,1]$ whose derivative is not computable.

But, by Cauchy's Theorem, differentiation of analytic functions is a computable operator!

First key idea:
If D is a rational disk whose boundary contains no zero of B, then we can compute

$$\frac{1}{2\pi i}\int_{\partial D}\frac{B'}{B}$$

exactly.



Problem:

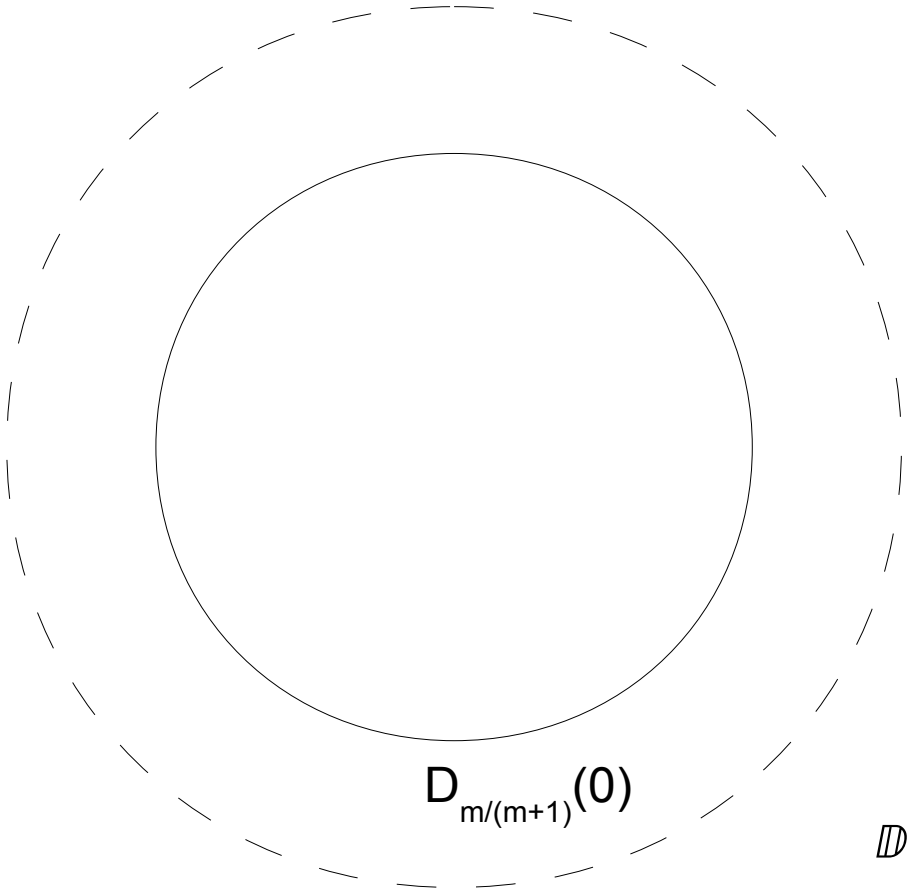
Proposition (M., unpublished)

There is a computable Blaschke product B for which no algorithm exists with the property that given any rational circle $C \subseteq \mathbb{D}$ as input determines if C contains a zero of B.

Solution: Use a 'wait and see' technique. That is, we can at least compute a list $C_1, C_2, \ldots$ of all rational circles $C \subseteq \mathbb{D}$ that contain no zero of B. From this, can compute a list $D_1, D_2, \ldots$ of all closed rational disks D contained in $\mathbb{D}$ such that ∂D contains a zero of B.

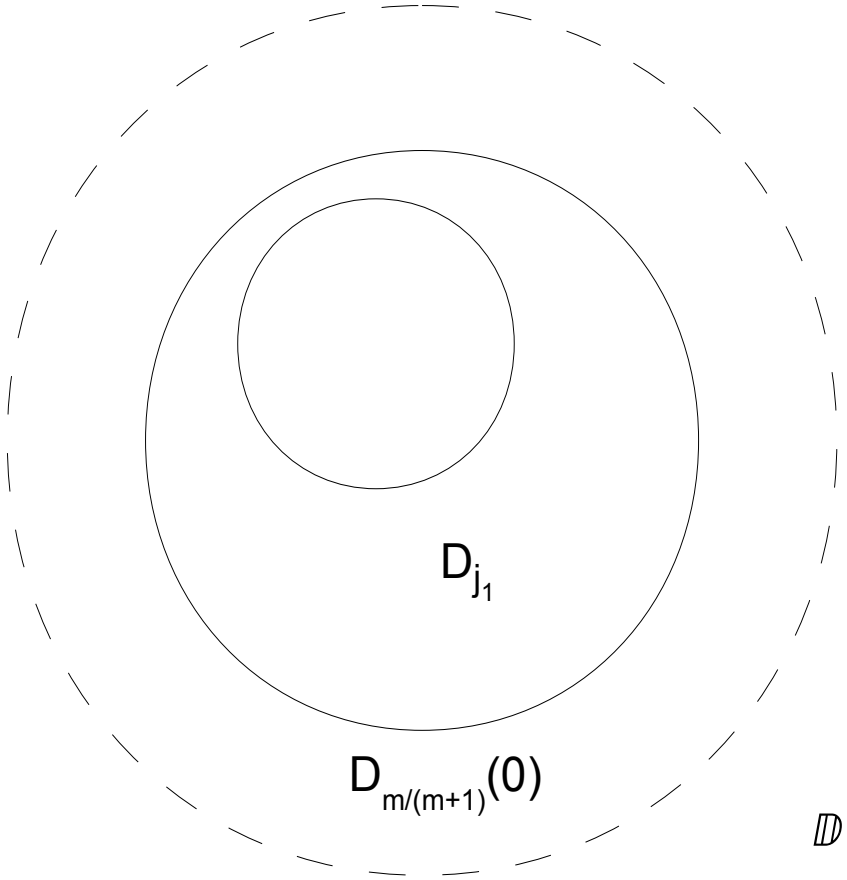
Definition of a_0:

Search for positive integer m such that $D_{m/(m+1)}(0)$ contains a zero of B:

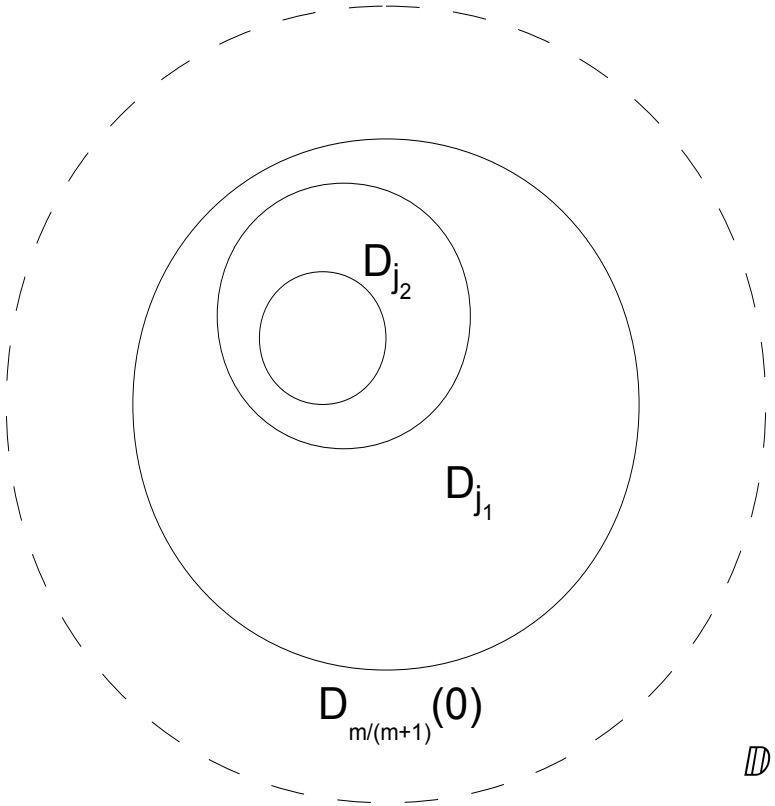


Defining a_0:

Search for an closed rational disk $D_{j_1} \subseteq D_{m/(m+1)}(0)$ that contains a zero of B and whose diameter is $< 1/2$.

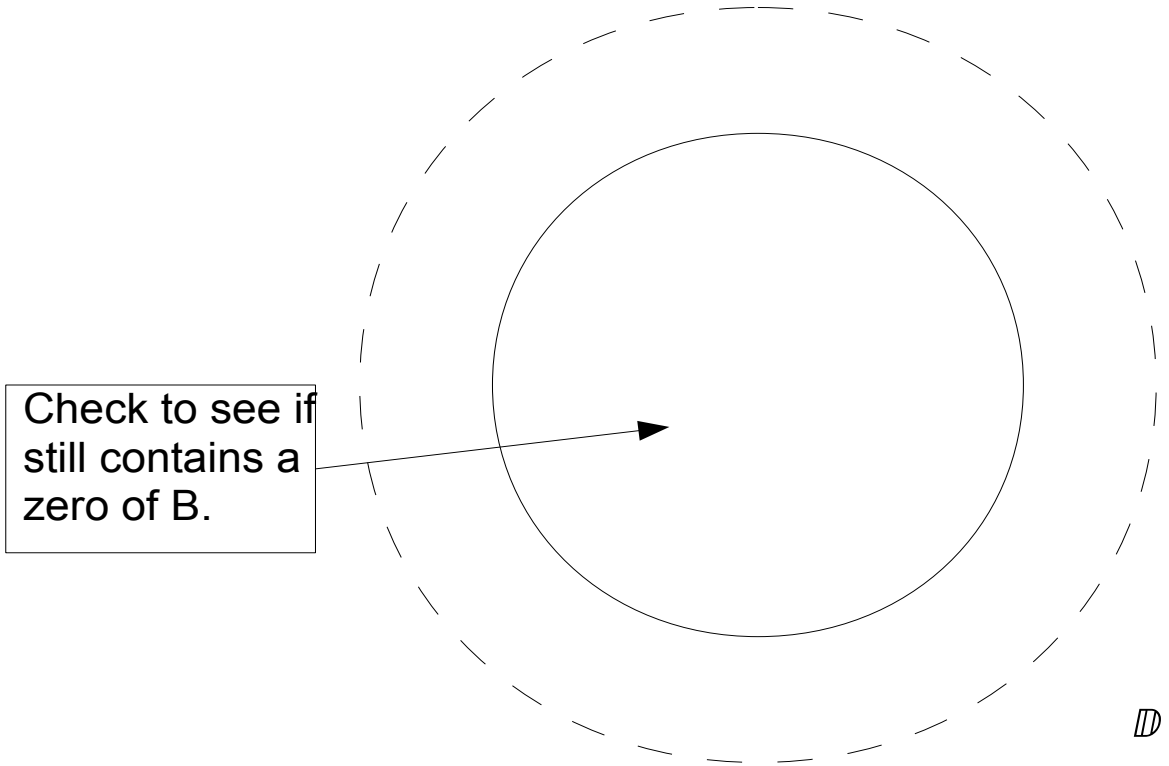


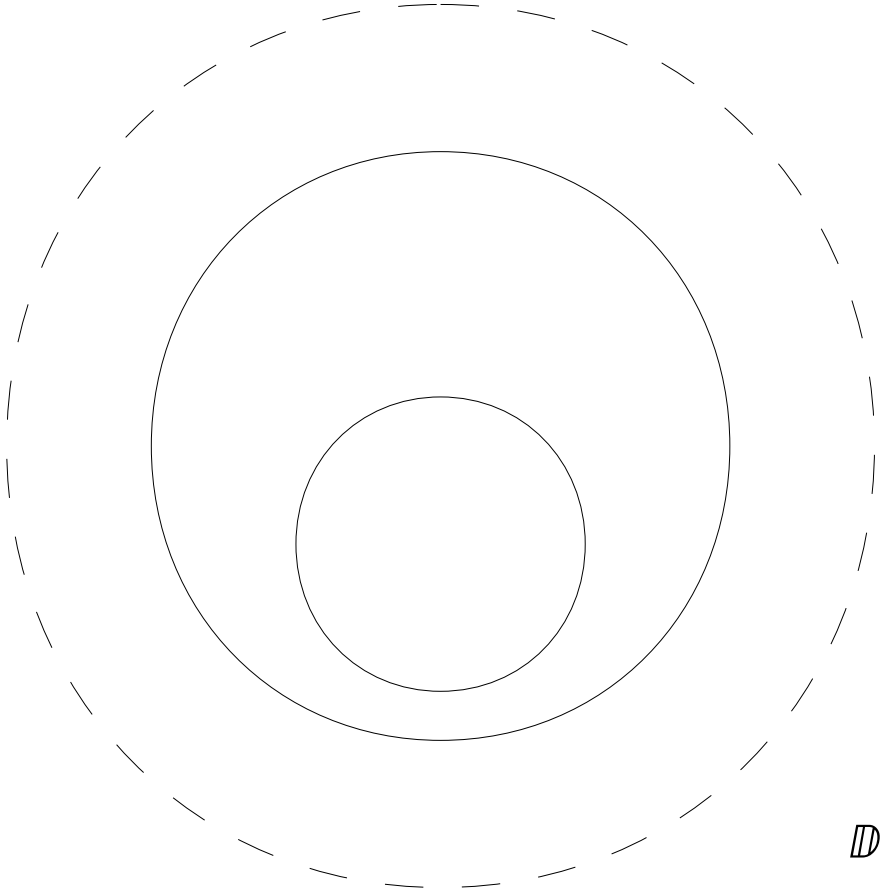
Then, search for an closed rational disk $D_{j_2} \subseteq D_{j_1}$ that contains a zero of B and whose diameter is $< 1/3$.

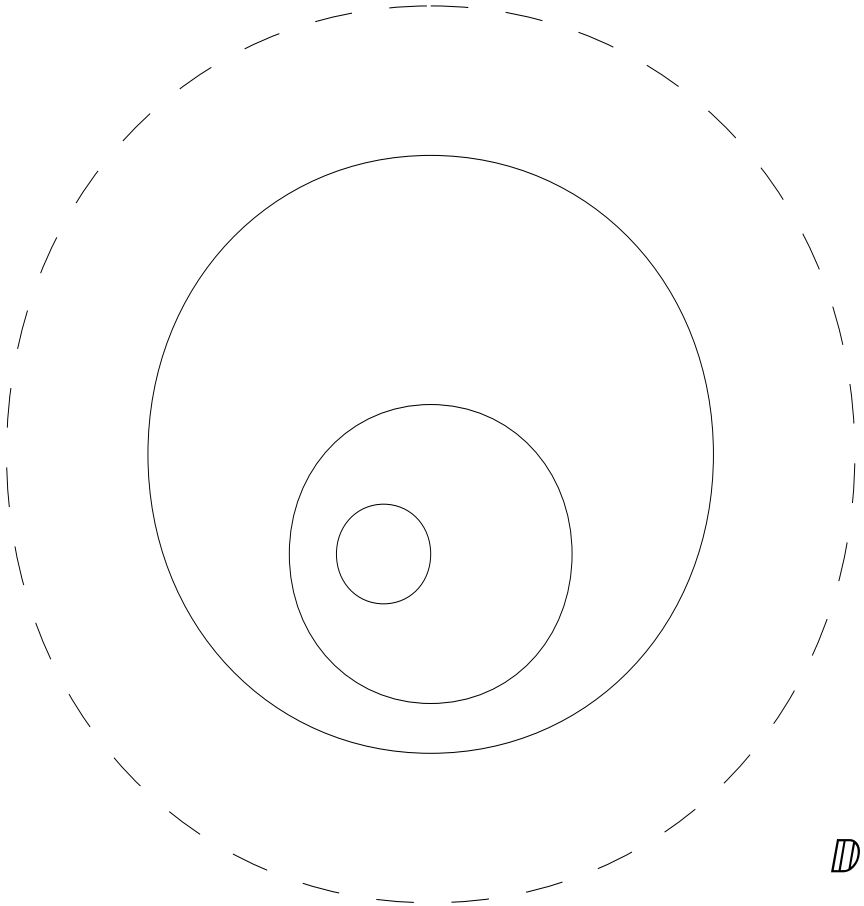


Keep doing this . . . Let a_0 be the unique point in $\bigcap_k D_{j_k}$.

Defining a_1:
Set $B := B/b_{a_1}$.

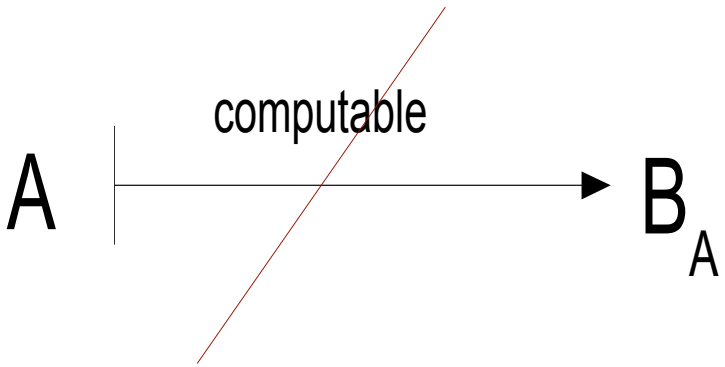


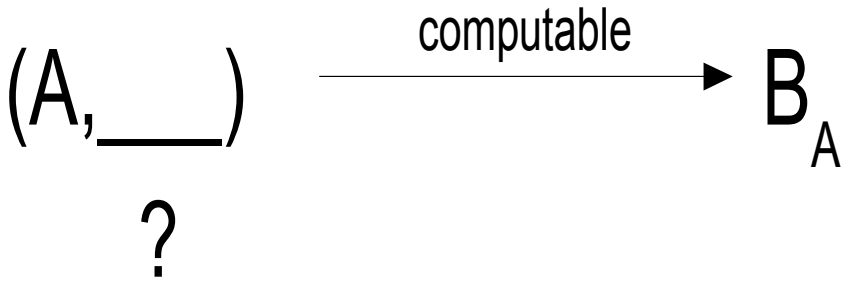

$\mathbb{D}$


$\mathbb{D}$

FACTS:

- $A =_{df} \{a_n\}_{n \in \mathbb{N}}$ is a computable sequence.
- The terms of A are precisely the zeros of B, and the number of times a zero is repeated in A is its multiplicity.


computable
A
B
A


computable
(A,___)
?
B
A

Let $\Sigma_A = \sum_{n=0}^{\infty} 1 - |a_n|$.

Theorem (M. 2008)

If A, Σ_A are computable, then B_A is computable.

Furthermore, Theorem holds in “highly uniform" way.

Theorem (M. 2008)

If A and $B_A(0)$ are computable, then Σ_A is computable.

Theorem (Caldwell, Pour-El 1975)

There is a computable sequence of complex numbers $\{a_n\}_{n=0}^{\infty}$ *such that:*

- $\sum_{n=0}^{\infty} a_n z^n$ *converges on* $\mathbb{C}$.
- *If* $f(z) = sum_{n=0}^{\infty} a_n z^n$, *then* f *is computable on every closed rational disk, but is not computable on* $\mathbb{C}$.

The remaining questions:

- Is there a computable, thin, and Blaschke sequence A such that B_A is not computable?
- Is there a computable Blaschke sequence A such that B_A is not computable and A satisfies the Frostman condition?
- If B is a computable Blaschke product, is its spectrum a computably closed/compact subset of ∂D?
- If E is a nowhere dense and computably closed subset of the boundary of $\mathbb{D}$, is there a computable Blaschke sequence/product with spectrum E?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