Computable presentations of metric structures

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Outline

Some history/background

- Beginning with the work of Malćev, Frölich, and Shepherdson, the field of computable structure theory has evolved into a significant branch of computability and logic in its own right that has generated numerous papers and challenging questions.
- Most of this work focused on countable structures from algebra and combinatorics and at times has even regarded uncountable structures as beyond its reach.
- But in 2013 that began to change.

Some history/background

- In a 2013 paper in JSL, A.G. Melnikov showed how the framework of computable analysis can be used to adapt the notions of computable structure theory to metric spaces.
- As observed by Franklin and McNicholl in 2020, this framework naturally extends to all metric structures.
- Today, I'll review results to date on metric spaces, Banach spaces, and C^* algebras as well as a number of open questions.
- Main themes: computable presentability, computable categoricity, degrees of categoricity.

Foundations: metric structures

Convention

Throughout this talk, $\mathbb{F}$ denotes the field of scalars. This can be either $\mathbb{R}$ or $\mathbb{C}$.

Definition

A *metric structure* is a sextuple $\mathcal{M} = (U, d, \mathcal{O}, \mathcal{F}, \eta)$ so that:

1. (U, d) is a complete metric space.
2. $\eta : \mathcal{O} \cup \mathcal{F} \to \mathbb{N}$.
3. Each $T \in \mathcal{O}$ is a uniformly continuous $\eta(T)$-ary operation on U.
4. Each $f \in \mathcal{F}$ is a uniformly continuous $\eta(f)$-ary functional on U; i.e. $f : U^{\eta(f)} \to \mathbb{F}$ and is uniformly continuous.

We treat constants as 0-ary operations. This is essentially the same definition used by model theorists. Main difference: for now we do not require the structure to be bounded.

Example

A complete metric space.

Example

Recall that a Banach space is a complete normed vector space over $\mathbb{F}$. Every Banach space $\mathcal{B} = (V, +, \cdot, \| \|, \mathbf{0})$ can be represented as a metric structure. Let:

- $U = V$
- $d(u, v) = \|u - v\|$.
- $\mathcal{O} = \{\mathbf{0}\} \cup \{+_{s,t} : s,t \text{ are scalars}\}$ where $+_{s,t}(u, v) = su + tv$.
- $\mathcal{F} = \{\| \|\}$.
- $\eta(\mathbf{0}) = 0$, $\eta(+_{s,t}) = 2$, $\eta(\| \|) = 1$.

Example

Similarly, every Hilbert space can be represented as a metric structure.

Example

Let $\Omega = (X, \mathcal{E}, P)$ be a probability space. Ω can be represented as a metric structure as follows.

- $U = \mathcal{E}$
- $d(A, B) = P(A \triangle B)$ (identify events whose symmetric difference is null).
- $\mathcal{O} = \{\emptyset, X, \cup, \cap, -\}$
- $\mathcal{F} = \{P\}$

Remark

If $\mathcal{M}_0 = (U_0, d_0, \mathcal{O}_0, \mathcal{F}_0, \eta_0)$ and $\mathcal{M}_1 = (U_1, d_1, \mathcal{O}_1, \mathcal{F}_1, \eta_1)$ are metric structures, then $\mathcal{M}_0 \times \mathcal{M}_1$ is defined in the obvious way. In particular, $d((p_0, p_1), (q_0, q_1)) = \max\{d_0(p_0, q_0), d_1(p_1, q_1)\}$.

Notation

When $S \subseteq U$, let $\langle S \rangle_{\mathcal{M}}$ denote the smallest subset of U that is closed under every operation in $\mathcal{O}$. $\langle S \rangle_{\mathcal{M}} =$ the algebra generated by S.

Foundations: presentations of metric structures

In order to define computability on metric structures, we adapt an idea from the seminal text of Pour-El and Richards.

Definition

1. A *structure* on $\mathcal{M}$ is a map $R : \mathbb{N} \to U$ so that $\overline{\langle \mathrm{ran}(R) \rangle_{\mathcal{M}}} = U$.
2. If R is a structure on $\mathcal{M}$, then $(\mathcal{M}, R)$ is a *presentation* of $\mathcal{M}$, and $R(n)$ is called the *n-th distinguished point* of $(\mathcal{M}, R)$.

Convention

We use $\mathcal{M}^{\#}$ and $\mathcal{M}^{+}$ to denote presentations of $\mathcal{M}$.

Remark

A presentation $\mathcal{M}^{\#}$ is completely specified by defining the enumeration of its distinguished points.

Foundations: presentations of metric structures

Remark

Presentations $\mathcal{M}_0^{\#}$ and $\mathcal{M}_1^{\#}$ of metric structures naturally induce a presentation $(\mathcal{M}_0 \times \mathcal{M}_1)^{\#}$ of $\mathcal{M}_0 \times \mathcal{M}_1$.

Remark

A metric structure may have a presentation that is designated as 'standard' by practice. In such a case, we identify the structure with its standard presentation.

Example

Consider the Banach space $\mathbb{F}$. The standard presentation of $\mathbb{F}$ is defined by declaring 1 to be the n-th distinguished point for all n.

Example

Consider the Banach space $C[0,1]$. The standard presentation of $C[0,1]$ is defined by declaring the n-th distinguished point to be $x \mapsto x^n$.

Definition

Suppose $\mathcal{M}^\#$ is a presentation of a metric structure $\mathcal{M}$.

1. The *rational points* of $\mathcal{M}^\#$ are the points in the algebra generated by the distinguished points of $\mathcal{M}^\#$.
2. A *rational open ball* of $\mathcal{M}^\#$ is an open ball of $\mathcal{M}$ whose center is a rational point of $\mathcal{M}^\#$ and whose radius is a positive rational number.

Example

The rational points of $C[0,1]$ are the polynomials over $\mathbb{F}$ with rational coefficients.

Foundations: numberings

Convention

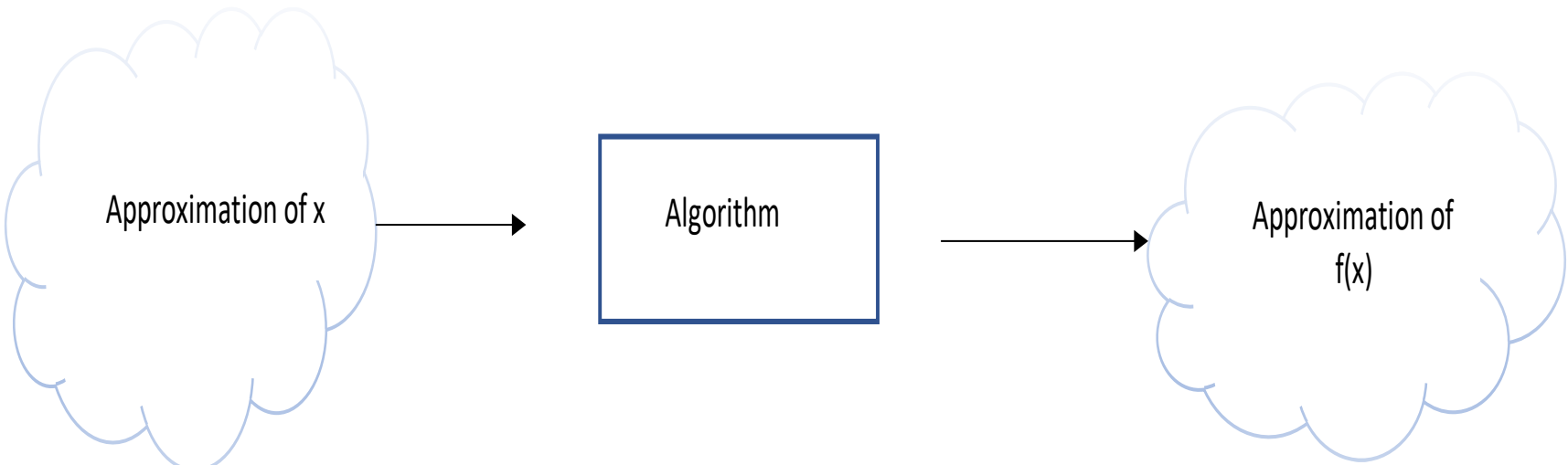
1. *From now on we will assume $\mathcal{O}$ and $\mathcal{F}$ are countable. (In the case of Banach spaces, it suffices to consider rational scalars only.) U may be uncountable.*
2. *We assume $\mathcal{O}$ and $\mathcal{F}$ have been numbered and we identify $\phi \in \mathcal{O} \cup \mathcal{F}$ with each number assigned to it.*
3. *Furthermore, we will assume $\mathcal{O}$ and $\mathcal{F}$ are computably numbered. That is:*
   1. *η is computable.*
   2. *There is a computable $g : (\mathcal{O} \cup \mathcal{F}) \times \mathbb{N} \to \mathbb{N}$ so that $g(\phi, \cdot)$ is a modulus of continuity for ϕ; e.g. if $T \in \mathcal{O}$ and $\eta(T) = 1$, then $d(T(u), T(v)) < 2^{-k}$ whenever $d(u, v) \leq 2^{-g(T,k)}$.*

Foundations: how to think about computing functions on metric structures

To define what we mean by a computable function in the metric setting requires us to rethink our assumptions about what it means to compute a function. We usually think of a computable function like this:

Foundations: how to think about computing functions on metric structures

However, a point in a metric structure typically requires an infinite amount of information to be specified, so this model no longer makes sense. In practice, one replaces exact specifications with approximations:



Foundations: how to think about computing functions on metric structures

- However, if our algorithm always produces the same approximation of $f(x)$, that's not very useful; we need a *convergence criterion*. That is, the machine can compute *arbitrarily good approximations* from *sufficiently good approximations*.
- Philosophy: an approximation is a rational open ball.
- There are numerous ways to formalize these intuitions; take your pick or come up with your own.

Foundations: computable presentations of metric structures

Definition

Suppose $\mathcal{M}^\#$ is a presentation of $\mathcal{M}$. We say $\mathcal{M}^\#$ is *computable* if

1. whenever p, q are rational points of $\mathcal{M}^\#$, $d(p, q)$ is computable uniformly in p, q, and
2. whenever f is an n-ary functional of $\mathcal{M}^\#$ and $p_1, \ldots, p_n$ are rational points of $\mathcal{M}^\#$, $f(p_1, \ldots, p_n)$ is computable uniformly in $f, p_1, \ldots, p_n$.

i.e. there is an algorithm that given $p_1, \ldots, p_n, q, f$ and $k \in \mathbb{N}$ computes rational $r, r' \in \mathbb{F}$ so that $|d(p_1, q) - r| < 2^{-k}$ and $|f(p_1, \ldots, p_n) - r'| < 2^{-k}$. An index of such an algorithm is referred to as an *index* of $\mathcal{M}^\#$.

Foundations: computable presentations of metric structures

Remark

- *The standard presentations of $\mathbb{F}$ and $C[0,1]$ are computable.*
- *If $\mathcal{M}_0^{\#}$ and $\mathcal{M}_1^{\#}$ are computable, then $(\mathcal{M}_0 \times \mathcal{M}_1)^{\#}$ is computable.*

Foundations: computable presentations of metric structures

Proposition

Suppose $\mathcal{M}^\#$ is a computable presentation of a metric structure $\mathcal{M}$.

1. *d is a computable map of $(\mathcal{M}^2)^\#$ into $\mathbb{F}$.*
2. *Each n-ary functional of $\mathcal{M}$ is a computable map of $(\mathcal{M}^n)^\#$ into $\mathbb{F}$.*
3. *Each n-ary operation of $\mathcal{M}$ is a computable map of $(\mathcal{M}^n)^\#$ into $\mathcal{M}^\#$*

Thus, a computable presentation of a metric structure provides all the information required to compute its operations and functionals.

Foundations: computable presentability and categoricity

Definition

A metric structure is *computably presentable* if it has a computable presentation.

Definition

A computably presentable metric structure is *computably categorical* if any two of its computable presentations are computably isometrically isomorphic.

Remark

Any notion of computable structure theory can be adapted to effective metric structure theory by replacing 'isomorphism' with 'isometric isomorphism'.

Outline

Lebesgue spaces

Most of our results on computable presentability are for Lebesgue spaces. Let's review what these are.

Definition

Let $\Omega = (X, \mathcal{S}, \mu)$ be a measure space, and suppose $1 \leq p \leq \infty$.

1. If $p < \infty$, then $L^p(\Omega) =$ set of all measurable $f : X \to \mathbb{F}$ so that $\int |f|^p \, d\mu < \infty$.
2. $L^\infty(\Omega) =$ set of all measurable $f : X \to \mathbb{R}$ for which there exist an M so that $|f(t)| < M$ for almost all $t \in X$.

Remark

1. *$L^p(\Omega)$ is a vector space under pointwise addition and scalar multiplication.*
2. *We identify functions in $L^p(\Omega)$ that agree a.e..*
3. *If $1 \leq p < \infty$, then $L^p(\Omega)$ is a Banach space under the norm*

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{1/p}.$$

Review of L^p spaces

Definition

1. An L^p *space* is a Banach space of the form $L^p(\Omega)$.
2. A Banach space is a *Lebesgue* space if it is an L^p space for some p.

Definition

1. $L^p[0,1] = L^p([0,1], \mathcal{S}, m)$ where $\mathcal{S}$ is the σ-algebra of Lebesgue measurable subsets of $[0,1]$ and $m =$ Lebesgue measure on $[0,1]$.
2. $\ell^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where μ is the counting measure on $\mathbb{N}$.
3. $\ell^p_n = L^p(\{1, \ldots, n\}, \mathcal{P}(\{1, \ldots, n\}), \mu)$ where μ is counting measure on $\{1, \ldots, n\}$.

Theorem (Classification of separable Lebesgue spaces)

Suppose $1 \leq p < \infty$. *Every separable nonzero* L^p *space is isometrically isomorphic to one of the following.*

1. ℓ^p_n
2. ℓ^p
3. $L^p[0,1]$
4. *An external direct sum of two of the above with* L^p*-norm. (i.e.* $\|(f,g)\|_p^p = \|f\|_p^p + \|g\|_p^p$*.)*

If $p \neq 2$, then no two of these spaces are isometrically isomorphic. If $p = 2$, the list reduces to ℓ^2 and ℓ^2_n.

Computable presentability of Lebesgue spaces

Theorem (M. 2020)

1. *If p is computable, then all separable L^p spaces are computably presentable.*
2. *If $n > 1$, and if ℓ^p_n is computably presentable, then p is computable.*
3. *If $p \geq 2$, and if there is a computably presentable L^p space whose dimension is at least 2, then p is computable.*
4. *If $1 \leq p < 2$, and if there is a computable presentable L^p space whose dimension is at least 2, then p is right-c.e..*

Conjecture

If $1 \leq p < 2$ is right c.e., then $L^p[0,1]$ is computably presentable.

Conjecture

If ℓ^p is computably presentable, then p is computable.

Computable presentability of $C(X)$ and $C^*(X)$

Definition

Suppose $X^\#$ is a computable presentation of a compact metric space X. $X^\#$ is *computably compact* if the set of all open finite covers of X by rational open balls is c.e..

Theorem (M., Fox 2022)

If X is homeomorphic to a space with a computably compact presentation, then $C(X)$, $C^*(X)$, and $C_0(X)$ are computably presentable.

Observation

The computable presentability of $C(X)$ does not imply the computable presentability of X. For, suppose X consists of exactly two points at an incomputable distance from each other. Then, X is not computably presentable but $C(X)$ is.

Theorem (Bazhenov, Harrison-Trainor, Melnikov 2022)

Suppose X is a Stone space. Then, $C(X)$ is computably presentable iff X is homemorphic to a metric space with a computably compact presentation.

Theorem (EFGMMMSHTT 2023+)

If $C^*(X)$ is computably presentable, then X is homeomorphic to a space with a computably compact presentation.

Theorem (Fox 2023+)

If $X \subseteq \mathbb{R}$ is compact, and if $C(X)$ is computably presentable, then X is homeomorphic to an arithmetically closed subset of $\mathbb{R}$.

Question

If $C(X)$ is computably presentable, does it follow that X is homeomorphic to a metric space with a computably compact presentation?

A connection with functional analysis

Definition

The *Kirchberg Embedding Problem (KEP)* is the statement that every C^* algebra embeds into an ultrapower of the Cuntz algebra $\mathcal{O}_2$.

Theorem (Goldbring, Hart 2023+)

If the Kirchberg Embedding Problem has a positive solution, then every computably weakly stable C^* algebra defined by finitely many $*$-polynomial equations over $\mathbb{Q}(i)$ is computably presentable.

Outline

Computable categoricity of metric spaces

Theorem (Melnikov 2013)

The following metric spaces are computably categorical.

1. *The Urysohn space.*
2. *The Cantor space.*
3. *Every separable Hilbert space.*

Theorem (Melnikov and Nies 2013)

1. *Every compact computably presentable metric space is Δ^0_3-categorical.*
2. *There is a compact computably presentable metric space that is Δ^0_3-categorical but not Δ^0_2-categorical.*

Observation

Every separable L^2 space is computably categorical.

Theorem

Suppose $p \in [1, \infty) - \{2\}$ is computable.

1. *If $n > 1$, then ℓ^p_n is computably categorical. (M. 2017)*
2. *The degree of categoricity of ℓ^p is $\mathbf{0}'$. (M. 2017)*
3. *$L^p[0,1]$ is computably categorical. (Clanin, M., Stull 2019)*
4. *The degree of categoricity of $\ell^p_n \oplus_p L^p[0,1]$ is $\mathbf{0}'$. (Brown, M. 2019)*
5. *The degree of categoricity of $\ell^p \oplus_p L^p[0,1]$ is $\mathbf{0}''$. (Brown, M. 2019)*

Computable categoricity of $C(X)$ and $C^*(X)$

Theorem (Melnikov, Ng 2013)

$C[0,1]$ is not computably categorical.

Theorem (Thewmorakot 2023)

$C(2^\omega)$ is not computably categorical.

Theorem (Fox 2023)

Every finite-dimensional C^* algebra is computably categorical.

Theorem (Fox 2023)

$C^*[0,1]$ is not computably categorical.

Theorem (Fox 2023+)

If $X \subseteq \mathbb{R}$ is a compact set that consists of the Cantor set together with finitely many points and closed intervals, then $C(X)$ is arithmetically categorical.

Outline

Index sets of classes of metric spaces

Theorem (Melnikov, Nies, 2013)

1. *The set of all indices of computable presentations of compact metric spaces is Π^0_3-complete.*
2. *The set of all (e, e') so that e, e' index computable presentations of isometric compact metric spaces is Π^0_2-complete within the set of indices of computable presentations of compact metric spaces.*

Theorem (Nies, Solecki 2015)

The set of all indices of computable presentations of locally compact Polish spaces is Π^1_1-complete.

Index sets of Lebesgue spaces

Theorem (Brown, M., Melnikov 2019)

If $p \geq 1$ is computable, then the set of all indices of computable presentations of L^p spaces is Π^0_2-complete.

Theorem (Brown, M., Melnikov 2019)

The set of all indices of computable presentations of Lebesgue spaces is Π^0_3.

Index sets of Lebesgue spaces

Theorem (Brown, M., Melnikov 2018+)

Suppose $p \geq 1$ is computable and not 2. Then, the set of all pairs (e, e') so that e and e' index computable presentations of isometrically isomorphic L^p spaces is co-3-Σ^0_3-complete.

Question

Is the set of all indices of computable presentations of Lebesgue spaces Π^0_3-complete?

Question

What is the complexity of the set of all pairs of natural numbers that index isometrically isomorphic Lebesgue spaces?

Banach spaces in general

Question

Is there a Δ^0_4-categorical Banach space that is not Δ^0_3-categorical?

Question

Is every finite-dimensional Banach space computably presentable?

Question

Is every computably presentable finite-dimensional Banach space computably categorical?

Outline

Transition to continuous logic

Changes in vocabulary:

- 'Functionals' $\rightarrow$ 'predicates'
- 'Operators' $\rightarrow$ 'functions'

Additional assumptions:

- Metric structures bounded (by 1).
- Predicates map into $[0, 1]$ ($0 =$ truth and $1 =$ falsehood).

- ‘$\forall$’ $\to$ ‘sup’
- ‘$\exists$’ $\to$ ‘inf’
- ‘$=$’ $\to$ ‘d’
- Connectives $\neg$, $\frac{1}{2}\times$, and $\dot{-}$
- Language $=$ signature of a metric structure. *Includes a modulus of continuity for each function and predicate symbol.*
- Fix a computably numbered language L (moduli of cont. computable uniformly). Assume all metric structures are interpretations of L.
- $\phi^{\mathcal{M}} =$ truth value of ϕ in metric structure $\mathcal{M}$.

Theorem (CGM 2021+)

Let $\mathcal{M}$ be a computably presentable L-structure.

1. $\{(\phi, r) : \phi$ *q.f.* $\wedge\ \phi^{\mathcal{M}} < r \in \mathbb{Q}^+\}$ *is* Σ^0_1.
2. $\{(\phi, r) : \phi$ *q.f.* $\wedge\ \phi^{\mathcal{M}} \leq r \in \mathbb{Q}^+\}$ *is* Π^0_1.
3. *For every positive integer N:*
   1. $\{(\phi, r) : \phi \in \Pi_N \wedge \phi^{\mathcal{M}} \leq r \in \mathbb{Q}^+\}$ *is* Π^0_N.
   2. $\{(\phi, r) : \phi \in \Pi_N \wedge \phi^{\mathcal{M}} < r \in \mathbb{Q}^+\}$ *is* Σ^0_{N+1}.
   3. $\{(\phi, r) : \phi \in \Sigma_N \wedge \phi^{\mathcal{M}} \leq r \in \mathbb{Q}^+\}$ *is* Σ^0_{N+1}.
   4. $\{(\phi, r) : \phi \in \Sigma_N \wedge \phi^{\mathcal{M}} < r \in \mathbb{Q}^+\}$ *is* Σ^0_N.

Let L_0 be the language that consists of countably many n-ary predicate symbols for each n.

Theorem (Camrud, Goldbring, M. 2023)

There is a computably presentable L_0-structure $\mathcal{M}_0$ so that:

1. $\{(\phi, r) : \phi$ *q.f.* $\wedge\ \phi^{\mathcal{M}} < r \in \mathbb{Q}^+\}$ *is* Σ^0_1*-complete.*
2. $\{(\phi, r) : \phi$ *q.f.* $\wedge\ \phi^{\mathcal{M}} \leq r \in \mathbb{Q}^+\}$ *is* Π^0_1*-complete.*
3. *For every positive integer N:*
   1. $\{(\phi, r) : \phi \in \Pi_N \wedge \phi^{\mathcal{M}} \leq r \in \mathbb{Q}^+\}$ *is* Π^0_N*-complete.*
   2. $\{(\phi, r) : \phi \in \Pi_N \wedge \phi^{\mathcal{M}} < r \in \mathbb{Q}^+\}$ *is* Σ^0_{N+1}*-complete.*
   3. $\{(\phi, r) : \phi \in \Sigma_N \wedge \phi^{\mathcal{M}} \leq r \in \mathbb{Q}^+\}$ *is* Σ^0_{N+1}*-complete.*
   4. $\{(\phi, r) : \phi \in \Sigma_N \wedge \phi^{\mathcal{M}} < r \in \mathbb{Q}^+\}$ *is* Σ^0_N*-complete*

Infinitary logic results

Theorem (CGM 2023)

Let $\mathcal{M}$ be a computably presentable L-structure and let ϕ be a computable infinitary sentence of L.

1. *If ϕ is Π^c_α, then $D^>(\phi^{\mathcal{M}})$ is $\Sigma^0_{\alpha+1}$ uniformly in a code of ϕ, and $D^\geq(\phi^{\mathcal{M}})$ is Π^0_α uniformly in a code of ϕ.*
2. *If ϕ is Σ^c_α, then $D^>(\phi^{\mathcal{M}})$ is Σ^0_α uniformly in a code of ϕ, and $D^\geq(\phi^{\mathcal{M}})$ is $\Pi^0_{\alpha+1}$ uniformly in a code of ϕ.*

Theorem

There is a language L'' and an L''-structure $\mathcal{M}$ so that the following hold for every computable ordinal α.

1. *There is a computable sequence $(\psi_i)_{i\in\mathbb{N}}$ of Π^c_α sentences of L'' so that $\{i : \frac{1}{2} \geq \psi_i^{\mathcal{M}}\}$ is Π^0_α-complete.*
2. *There is a computable sequence $(\psi_i)_{i\in\mathbb{N}}$ of Σ^c_α sentences of L'' so that $\{i : \frac{1}{2} > \psi_i^{\mathcal{M}}\}$ is Σ^0_α-complete.*
3. *There is a computable sequence $(\psi_i)_{i\in\mathbb{N}}$ of Π^c_α sentences of L'' so that $\{i : \frac{1}{2} > \psi_i^{\mathcal{M}}\}$ is $\Sigma^0_{\alpha+1}$-complete.*
4. *There is a computable sequence $(\psi_i)_{i\in\mathbb{N}}$ of Σ^c_α sentences of L'' so that $\{i : \frac{1}{2} \geq \psi_i^{\mathcal{M}}\}$ is $\Pi^0_{\alpha+1}$-complete.*

Definition

Let $\mathcal{M}$ be an L-structure.

1. $\mathrm{Th}(\mathcal{M}) : \mathrm{Sent}_L \to [0,1]$ defined by $\mathrm{Th}(\mathcal{M})(\phi) = \phi^{\mathcal{M}}$.
2. $\mathrm{Th}_{\forall_n}(\mathcal{M})$ is restriction of $\mathrm{Th}(\mathcal{M})$ to the Π_n sentences of L.

Corollary

If $\mathcal{M}$ is computably presentable, then:

1. $\emptyset^{(\omega)}$ *computes* $\mathrm{Th}(\mathcal{M})$.
2. *For each positive n,* $\emptyset^{(n)}$ *computes* $\mathrm{Th}_{\forall_n}(\mathcal{M})$.

Corollary (Goldbring-Hart, 2020)

If $\mathcal{M} \in \{C(2^\omega), C(\mathbb{P})\}$, *then :*

1. $\emptyset^{(\omega)}$ *computes* $\mathrm{Th}(\mathcal{M})$.
2. *For each positive integer n,* $\emptyset^{(n)}$ *computes* $\mathrm{Th}_{\forall_n}(\mathcal{M})$.

Observation

1. $\emptyset^{(\omega)}$ *is the Turing degree of* $\mathrm{Th}(\mathcal{M}_0)$. *(That is, if X computes* $\mathrm{Th}(\mathcal{M}_0)$*, then X computes* $\emptyset^{(\omega)}$*.)*
2. *For each positive integer n,* $\emptyset^{(n)}$ *is the Turing degree of* $\mathrm{Th}_{\forall_n}(\mathcal{M}_0)$.

Let $\mathcal{R}$ denote the hyperfinite II_1 factor.

Theorem (Goldbring, Hart 2020+)

The Turing degree of $\mathrm{Th}_{\forall_1}(\mathcal{R})$ *is* $\emptyset'$.

Proof use technology from proof of $\mathrm{MIP}^* = \mathrm{RE}$ result.

Question

If $\mathcal{M}$ is a computably presentable L-structure, does $\mathrm{Th}_{\forall_n}(\mathcal{M})$ have a Turing degree? That is is there a least Turing degree that computes $\mathrm{Th}_{\forall_n}(\mathcal{M})$?

Question

For $n \geq 2$, does $\mathrm{Th}_{\forall_n}(\mathcal{R})$ have a Turing degree and if so is it $\emptyset^{(n)}$?