An exploration of effective local connectivity

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For the past three years I've been thinking a lot about the following two things:

- Images of the unit interval under computable maps
- Computably extending the domain of conformal maps

There is an issue that arises in these investigations: effective local connectivity. I'd like to share with you what this concept is and how it naturally arises in these considerations. It also leads to some interesting problems which I also wish to share.

In particular, I'd like us to think about the effective content of the following theorems of classical topology and analysis:

- **Hahn-Mazurkiewicz Theorem**: Let X be a non-empty Hausdorff space. Then, there is a continuous surjection of the unit interval onto X if and only if X is compact, connected, metrizable, and locally connected.
- **Carathéodory Theorem**: If ϕ is a conformal map of the unit disk onto a Jordan domain D, then ϕ extends to a homeomorphism of the closed unit disk with $\overline{D}$.
- **Continuity Theorem**: If ϕ is a conformal map of the unit disk onto a bounded domain D, then ϕ has a boundary extension if and only if ∂D is locally connected.

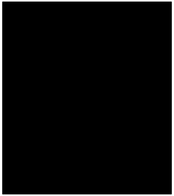
If you are unfamiliar with some of the terminology in these theorems don't worry, I will define everything as we go along.

So, let's start with the Hahn-Mazurkiewicz Theorem. To motivate this theorem, let's begin with a 'joke':

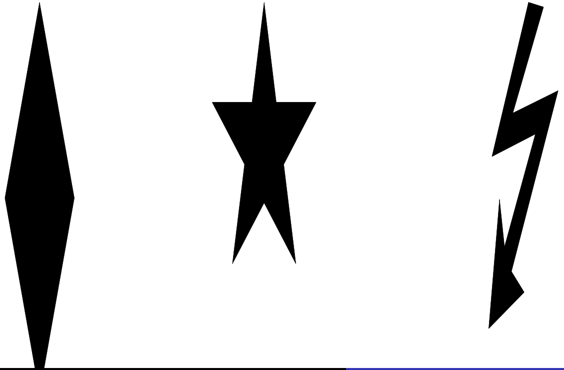
Physics definition

A surface is one-dimensional if it is the image of a continuous function on the unit interval.

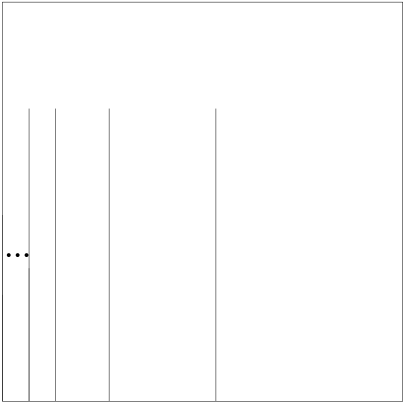
Of course, we all know that this is a bad definition:

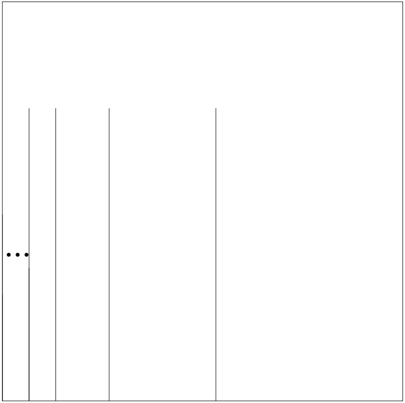


There are a lot of other spaces that are images of the unit interval under a continuous function. Here are a few:

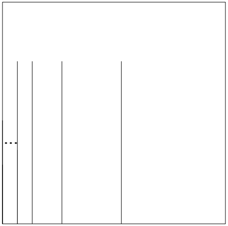


So which spaces are images of continuous maps on the unit interval? Trivially, any such space must be compact and connected. But, this isn't the whole story. *e.g.* the 'comb space':





It is ‘well-known’ that there is no continuous map of the unit interval onto the comb space. I’d like to sketch a basic proof of this which I think will be instructive.



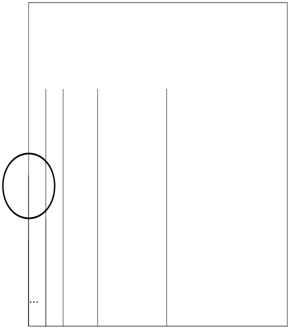
Proof sketch: Suppose f is a continuous surjection of the unit interval onto the comb space. Even though f is not injective, f has an inverse modulus of continuity; that is when points in its range are sufficiently close together, they have pre-images that are as close as we like. So, take two tips that are very close together so that they have pre-images that are very close. The interval between these pre-images then traverses two comb teeth. This violates uniform continuity of f.

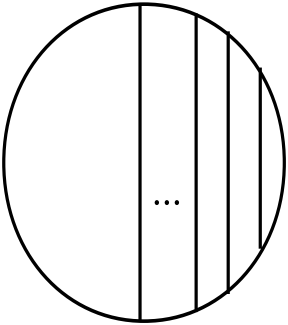
So what is the additional property that we need (besides compactness and connectedness)? It turns out that local connectedness provides the missing link. Here is a definition of this property.

Definition

A topological space X is *locally connected* if whenever U is a neighborhood of a point p in X, U contains a connected neighborhood of p.

It is easy to see that the comb space is not effectively locally connected:





These observations lead us to the Hahn-Mazurkiewicz Theorem:

Theorem (Hahn-Mazurkiewicz Theorem)

A non-empty Hausdorff space is the image of a continuous function on the unit interval if and only if it is compact, connected, metrizable, and locally connected.

So, a non-empty subset of $\mathbb{R}^n$ is the image of a continuous function on the unit interval if and only if it is compact, connected, and locally connected.

Now, let's think about the effective content of this theorem.

First, let's define a notion of computable compactness.

Definition

$\emptyset \neq X \subseteq \mathbb{R}^n$ is *computably compact* if it is compact and if there is an algorithm that given any $k \in \mathbb{N}$ as input produces a finite set of rational boxes $\{B_1, \ldots, B_t\}$ such that

- $\mathrm{diam}(B_j) < 2^{-k}$,
- $X \cap B_j \neq \emptyset$, and
- $X \subseteq \bigcup_j B_j$.

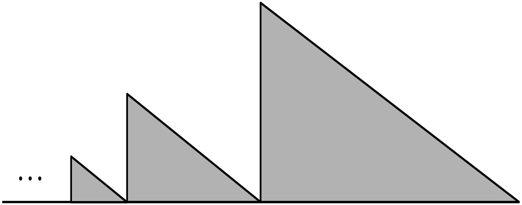
When $n = 2$, this is equivalent to being able to plot the set on a computer screen at arbitrarily good resolution.

It's 'easy' to show that computable compactness is not good enough:

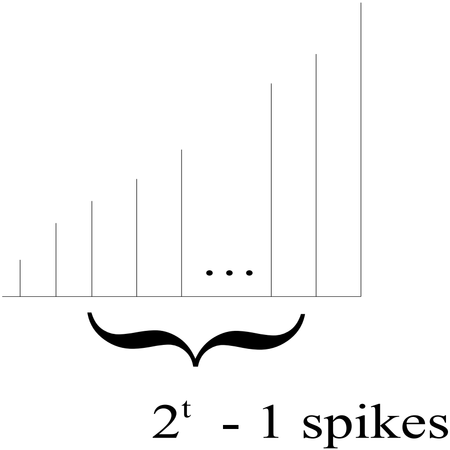
Theorem (Couch, Daniel, M. 2012)

There is a computably compact $X \subseteq \mathbb{R}^2$ that is the image of a continuous function on the unit interval but not the image of any <u>*computable*</u> *function on the unit interval.*

Proof sketch: Fix an incomputable *c.e.* set A. Start with



If n enters A at stage t, replace $(n+1)$-st triangle with:



Clearly X is connected, locally connected, and computably compact. So, it is the image of a continuous map on the unit interval. But, it is not the image of a *computable* map on interval.

Proof is like proof that comb space is not the image of a continuous map on the unit interval except that one uses a computable modulus of continuity for f instead of uniform continuity. The existence of a computable surjection of the unit interval onto X leads to a decision procedure for A.

This isn't really surprising: we didn't include an effective version of the local connectivity hypothesis! How should this be defined?

Here are two definitions:

Definition (Brattka 2008)

$X \subseteq \mathbb{R}^n$ is *effectively locally connected* if there is a type two machine M such that whenever a name of a point $p \in X$ and a name of an open $U \subseteq \mathbb{R}^n$ that contains p are written on the input tapes, and M is allowed to run indefinitely, a name of an open $V \subseteq \mathbb{R}^n$ is written on the output tape so that $p \in V \subseteq U$ and $V \cap X$ is connected.

Definition (J. Miller 2002)

A compact set $X \subseteq \mathbb{R}^n$ is *effectively locally connected* if there is a computable $f : \mathbb{N} \to \mathbb{N}$ such that whenever $k \in \mathbb{N}$ and $p \in X$, there is a connected set $C \subseteq X$ such that

$$B_{2^{-f(k)}}(p) \cap X \subseteq C \subseteq B_{2^{-k}}(p)$$

I'd like to propose one more definition.

First a motivating observation: two points in a topological space may be close, but you may have to travel a long way from the first point to get to the second point:

e.g. when the shortest route to your office is blocked by construction. This leads to the following definition:

Definition

Let $X \subseteq \mathbb{R}^n$. A *modulus of local connectivity* for X is a function $g : \mathbb{N} \to \mathbb{N}$ so that for all $k \in \mathbb{N}$, whenever $p, q \in X$ are such that $0 < d(p, q) \leq 2^{-g(k)}$, X includes an arc from p to q whose diameter is smaller than 2^{-k}.

So, roughly speaking, a modulus of local connectivity for X predicts how close two points have to be in X so that one can travel from the first point to the second without going to far from the first.

Here is the definition of 'effective local connectivity' I'd like to propose:

Definition

Suppose $X \subseteq \mathbb{R}^n$ is compact. X is *effectively locally connected* if it has a computable modulus of local connectivity.

Notes:

- Miller's definition is an adaptation of the classical notion of 'connected *im kleinen*'.
- The modulus of local connectivity definition is an adaptation of the classical notion of 'uniformly locally arcwise connected'.
- Because of this, moduli of local connectivity were called ULAC functions in some of my earlier papers on these topics. Since then I've changed the terminology.

One more note:

- Every subset of $\mathbb{R}^n$ that is non-empty, compact, connected, and locally connected has a modulus of local connectivity; just not necessarily a computable one!

Theorem (essentially in Daniel, M. 2012)

All three of these definitions of 'effectively locally connected' are equivalent for subsets of $\mathbb{R}^n$ that are connected and computably compact.

Notes:

- The proof is 'ultra-uniform'.
- Only the first definition (Brattka's) applies to non-compact spaces. However, in most applications the spaces considered are computably compact; in this case I think the modulus of local connectivity definition gives the most information most quickly.
- Effective local connectivity implies local connectivity.

So, now we get an effective version of the first direction of the Hahn-Mazurkiewicz Theorem:

Theorem (Couch, Daniel, M. 2012)

Suppose X is a subset of $\mathbb{R}^n$ that is connected and computably compact. If X has a computable modulus of local connectivity, then there is a computable map of the unit interval onto X.

I won't go through the entire proof of this theorem, but I'd like to at least sketch the ideas involved and how the modulus of local connectivity is used:

Proof sketch:

- Let $C =$ Cantor's middle third set. Start by computing a surjection of C onto X (using computable compactness); label this surjection f. Want to extend f to the entire unit interval.
- Fill in the values of f on removed intervals with arcs that join images of endpoints. To ensure the continuity of f, the diameters of these arcs need to $\to 0$ as the lengths of the intervals $\to 0$ AND we need this to happen computably.
- Using the modulus of local connectivity, we determine how small a diameter is possible based on length of interval.
- We then use a "search procedure" to "find" these arcs.

- Again proof is ‘ultra-uniform’.
- What about converse?
- Answer: the converse fails effectively. In fact, counterexamples abound!

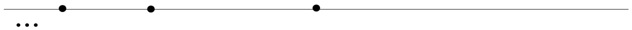
These counterexamples as based on following:

Theorem (Miller 2002, Daniel, M. 2012)

Suppose $\sigma \subseteq \mathbb{R}^n$ is a computably compact arc. Then, there is a computable homeomorphism of the unit interval with σ if and only if σ has a computable modulus of local connectivity.

Here is a very simple counterexample (essentially due to J. Miller): take an incomputable *c.e.* set A.

Start with:



(Line segment is the unit interval; the dots are at powers of $1/2$.)

If n enters A at t replace the n-th line segment with a narrow zigzag (e.g. diameter $< 2^{-t}$):

A computable modulus of local connectivity for this space yields a decision procedure for A.

Note that there is a computable surjection of $[0, 1]$ onto this space: start by retracing each interval.

There is an even worse counterexample:

Theorem (Gu, Lutz, Mayordomo 2009)

There is an arc $\sigma \subseteq \mathbb{R}^2$ so that there is a computable map of the unit interval onto σ BUT any such map must 'retrace itself infinitely often'.

This is worse because the previous example is at least 'locally effectively locally connected'.

In any case, the upshot of these results is that effective local connectivity is providing too much information for the effective Hahn-Mazurkiewicz Theorem. This leads to the following:

Problem

What is the right additional parameter for the effective Hahn-Mazurkiewicz Theorem?

In addition:

Problem

Describe the computable homeomorphism types of the arcs that do not have a computable modulus of local connectivity even though they are computable images of the unit interval. e.g. what are some invariants for these computable homeomorphism classes?

Let's move on now to the Carathéodory Theorem. Since this is an extension of the Riemann Mapping Theorem, let's start by reviewing the Riemann Mapping Theorem. Let's first go over some preliminary terminology:

Recall that a *domain* is a subset of $\mathbb{C}$ that is non-empty, open, and connected.

The *unit disk* is $\{z \in \mathbb{C} \ : \ |z| < 1\}$. The *closed unit disk* is $\{z \in \mathbb{C} \ : \ |z| \leq 1\}$. The *unit circle* is $\{z \in \mathbb{C} \ : \ |z| = 1\}$.

Theorem (Riemann Mapping Theorem)

If $D \neq \mathbb{C}$ is a simply connected domain, then there is a conformal map of the unit disk onto D.

Theorem (Carathéodory Theorem)

If the boundary of D is a Jordan curve, and if ϕ is a conformal map of the unit disk onto D, then ϕ extends to a homeomorphism of the closed unit disk with $\overline{D}$.

This extension is an example of what is called a *boundary extension*. i.e. a continuous extension of ϕ to the closed unit disk.

Convention

If ϕ denotes a conformal map that has a boundary extension, then this boundary extension is also denoted by ϕ.

So, if ϕ is a computable map of the unit disk onto a domain D whose boundary is a Jordan curve, what information is required to compute the boundary extension of ϕ? Our first result is negative:

Theorem (M. 2012)

There is a computable conformal map of the unit disk onto a Jordan domain whose boundary extension is incomputable.

Proof sketch: Take the 'zig-zagging' example above and wrap it around a circle. Then, make use of the following:

Proposition (M. 2012)

If f is a computable homeomorphism of the unit circle with X, then X has a computable modulus of local connectivity.

It's obvious what's missing:

Theorem (M. 2012)

Suppose ϕ is a computable conformal map of the unit disk onto a Jordan domain D. Then the boundary extension of ϕ is computable if and only if the boundary of D has a computable modulus of local connectivity.

Again, the proof is ultra-uniform in both directions.

So, this time the modulus of local connectivity is both a necessary and sufficient additional parameter.

Lately I've been working on a better proof of this theorem based on the following. Here, we work with the inverse map from the domain onto the unit disk.

When ψ is a conformal map of a domain D onto the unit disk and $\zeta_0 \in \partial D$, let:

$$\mu_\epsilon(\zeta_0) = \min\left\{ |\zeta_0 - \psi^{-1}(w)| \ : \ |w| \leq \sqrt{(1-l)^2 + \frac{\epsilon^2 - l^2}{4}} \right\}$$

$$\Delta_\epsilon(\zeta_0) = \sup_{0<l<\epsilon} \left\{ \exp\left(\frac{8\pi^2}{l^2 - \epsilon^2}\right) \mu_\epsilon(\zeta_0) \right\}.$$

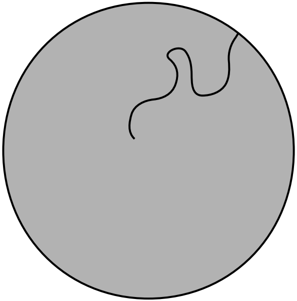
Theorem (M. 2014)

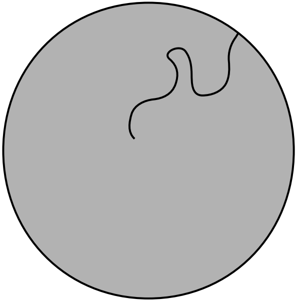
Suppose ψ is a conformal map of a Jordan domain D onto the unit disk. Let ζ_0 be a boundary point of D, and let g be a modulus of local connectivity for the boundary of D. Then, for each $\epsilon > 0$, $|\psi(z_0) - \psi(\zeta_0)| < \epsilon$ whenever z_0 is a point in D so that $|z_0 - \zeta_0| < 2^{-g(k)}$ and k is a non-negative integer so that $2^{-k} + 2^{-g(k)}$ is smaller than $\Delta_\epsilon(\zeta_0)$.

Key take-away from this: using a modulus of local connectivity for the boundary of a Jordan domain, we can determine how close $z_0 \in D$ must be to $\zeta_0 \in \partial D$ in order for $|\psi(z_0) - \psi(\zeta_0)|$ to be smaller than ϵ. It then follows that the extension of ψ to $\overline{D}$ is computable if ψ is.

Finally, let's consider the effective content of the Continuity Theorem. I will first motivate statement of this theorem by discussing boundary extensions in general.

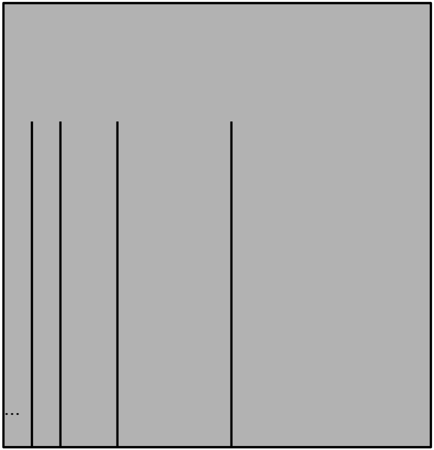
Sometimes boundary extensions exist even when the target domain is not Jordan. For example, let D be the domain:

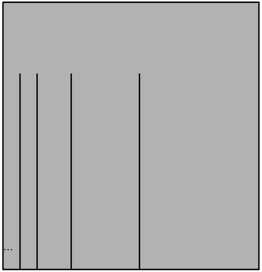




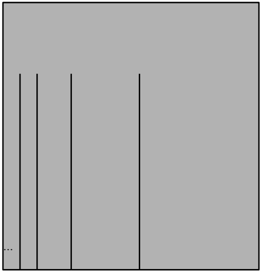
By the Riemann Mapping Theorem there is a conformal map of the unit disk onto D. Any conformal map of the unit disk onto D extends to a *continuous* (but not one-to-one!) map of the closed unit disk onto $\overline{D}$. Each point on the slit has exactly two pre-images under such a map.

But, sometimes boundary extensions don't exist. For example, let D be the interior of the comb space:

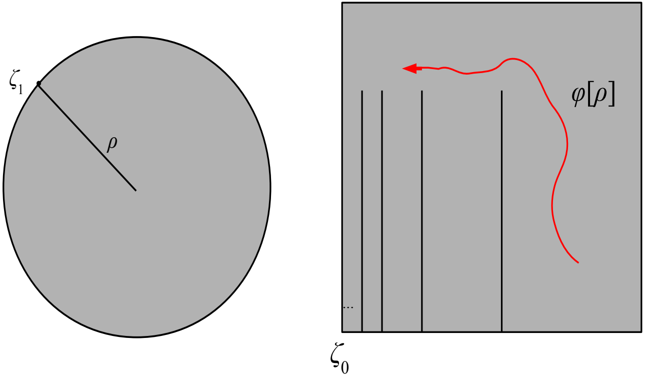




By the Riemann Mapping Theorem, there is a conformal map of the unit disk onto D. However, such a map can not have a boundary extension. *Quick Proof:* Otherwise there is a continuous map of the unit interval onto the comb space.



Sketch of another proof: Suppose ϕ is a conformal map of the unit disk onto D that extends to a continuous map of the closed unit disk onto $\overline{D}$.



Then ϕ maps ρ onto an *accessing arc* for ζ_0. That is, an arc from $\phi(0)$ to ζ_0 that contains no boundary point of D except ζ_0. This is impossible.

Theorem (Continuity Theorem)

Suppose ϕ is a conformal map of the unit disk onto a bounded domain D. Then, ϕ has a boundary extension if and only if the boundary of D is locally connected.

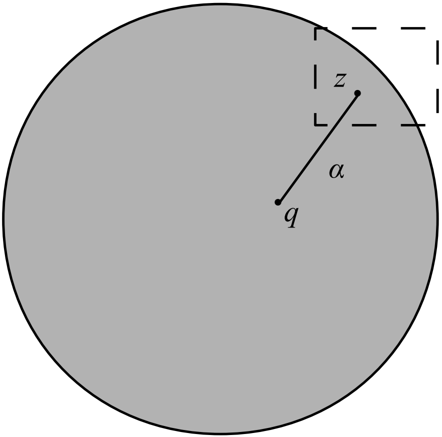
The direction ($\Leftarrow$) is 'trivial' by the Hahn-Mazurkiewicz Theorem.

One direction of this is effectively true:

Theorem (M. 2014)

Suppose ϕ is a computable conformal map of the unit disk onto a bounded domain D. If the boundary of D has a computable modulus of local connectivity, then the boundary extension of ϕ is computable.

Again, I won't go through the entire proof of this theorem, but I'd like to sketch a few of the steps and give you an idea of how the modulus of local connectivity is used. Here is a proof sketch (literally):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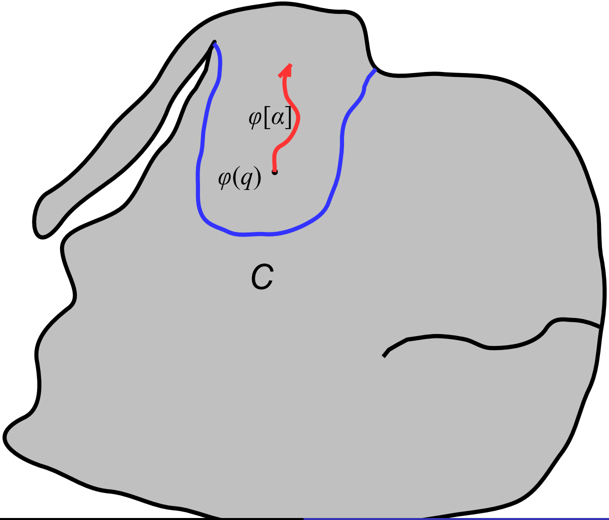
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Again, proof is ‘ultra-uniform’.

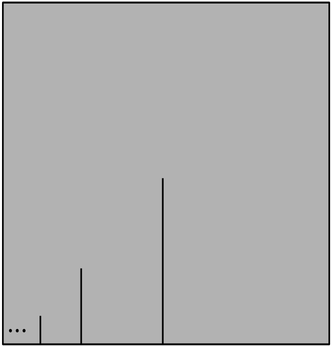
What about the converse? On the one hand, the converse of the Hahn-Mazurkiewicz Theorem failed to be true effectively. On the other hand, perhaps the conformality of ϕ might help us...

Or not:

Theorem (M. 2014)

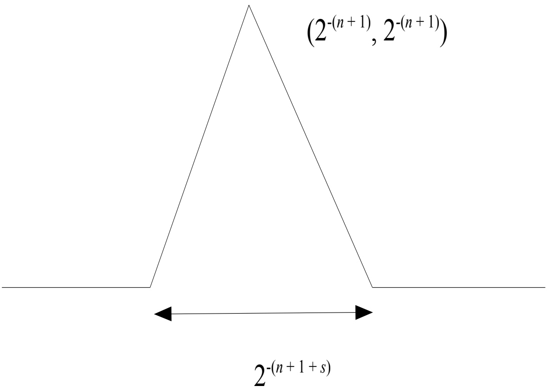
There is a computable conformal map of the unit disk onto a domain D so that ϕ has a computable boundary extension even though the boundary of D does not have a computable modulus of local connectivity.

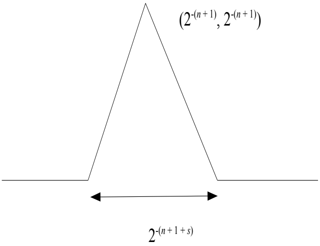
Proof sketch: We build D as follows. Start with:



A spike at each power of $\frac{1}{2}$. Each spike goes from bottom of square to diagonal.

If n enters A at stage s, replace spike at $2^{-(n+1)}$ with a tent:





So, it follows that the boundary of D does not have a computable modulus of local connectivity. (*Proof sketch:* If it did, we could use a modulus of local connectivity to decide if $n \in A$; if $n \in A$ then there are points close together so that one must travel relatively far from one point to get to the other.)

On the other hand: by Hertling's Effective Riemann Mapping Theorem there is a computable conformal map of the unit disk onto *D*, ϕ. The boundary of *D* is locally connected, so ϕ has a boundary extension.

The geometry of the boundary of *D* allows us to compute the boundary extension of ϕ even though the boundary of *D* does not have a computable modulus of local connectivity.

In other words, effective local connectivity once again provides sufficient and non-superfluous information, but not necessary information.

Problem

What is the right additional parameter for an effective version of the Continuity Theorem?

- Three natural definitions of ‘effective local connectivity’ are equivalent (including the most direct translation into TTE).
- Using a modulus of local connectivity provides a very useful definition of ‘effective local connectivity’.
- Effective local connectivity arises naturally in several problems in computable analysis by controlling ‘zig-zagging’.
- Nevertheless, it often provides *too much* information. What are the right parameters?

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