Isometry Degrees of Computable Copies of ℓ^p

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Motivation

- Traditionally, computable structure theory has focused on countable structures (e.g. countable graphs, groups, linear orders).
- This makes sense. By means of codings, we can establish a reasonable computability theory on almost any countable structure.
- But, we now have a better understanding of how to set up a computability theory on an uncountable structures; e.g. computable metric spaces.
- So, why bring computable structure theory into the realm of computable analysis?

- Uncountable structures such as metric spaces and Banach spaces are common in analysis.
- So, it makes sense to apply the concepts of computable structure theory (such as computable categoricity) to classes of uncountable spaces such as metric spaces and Banach spaces.
- Recently Greenberg, Melnikov, Ng, Nies, Turetsky have obtained a number of interesting results about polish spaces.
- Today we will discuss some recent results on a class of Banach spaces: the ℓ^p spaces.
- n.b. in this talk all Banach spaces are Banach spaces over $\mathbb{C}$.

Review of ℓ^p

Suppose $0 < p < \infty$.

- Recall that ℓ^p is the set of all sequences of complex numbers $\{a_n\}_{n=0}^{\infty}$ so that

$$\sum_{n=0}^{\infty} |a_n|^p < \infty.$$

- e.g. ℓ^1 is the set of all absolutely convergent sequences. ℓ^2 is the set of all square summable sequences.
- If $p \geq 1$, ℓ^p is a Banach space under the norm

$$\|\{a_n\}_n\|_p = \left(\sum_{n=0}^{\infty} |a_n|^p\right)^{1/p}.$$

Computability on Banach spaces: generating sets

Suppose $\mathcal{B} = (V, +, \cdot, \| \|)$ is a Banach space. Let $S \subseteq V$.

- S is a *generating set* for $\mathcal{B}$ if V is the closed linear span of S.
- Example: Let $e_n = \chi_{\{n\}}$. Let $E = \{e_n \ : \ n \in \mathbb{N}\}$. Then, E is a generating set for ℓ^p. We call E the *standard generating set* for ℓ^p. $E =$ default generating set.

Computability on Banach spaces: effective generating sets

Suppose $\mathcal{B} = (V, +, \cdot, \| \|)$ is a Banach space.

- When K is a subfield of $\mathbb{C}$, the *span of S over K* is

$$\mathcal{L}_K(S) = \left\{ \sum_{j=0}^{M} \alpha_j v_j \ : \ \alpha_0, \ldots, \alpha_M \in K \ \wedge \ v_0, \ldots, v_M \in S \right\}$$

Definition 1

Let $F = \{f_0, f_1, \ldots\}$ be a generating set for $\mathcal{B}$. F is an *effective generating set* if there is an algorithm that given $k \in \mathbb{N}$ and $f \in \mathcal{L}_{\mathbb{Q}(i)}(F)$ computes $q \in \mathbb{Q}$ so that $|q - \|f\|| < 2^{-k}$. i.e. if $\| \|$ is a computable function on $\mathcal{L}_{\mathbb{Q}(i)}(F)$.

Some examples of effective generating sets

Suppose p is a computable real so that $p \geq 1$.

- The standard generating set for ℓ^p is an effective generating set for ℓ^p.
- If ζ is unimodular (i.e. $|\zeta| = 1$), then $\zeta E := \{\zeta e_n \ : \ n \in \mathbb{N}\}$ is an effective generating set for ℓ^p.

n.b. ℓ^p has an effective generating set if and only if p is computable.

Computability via effective generating sets

Suppose F is an effective generating set for $\mathcal{B}$.

- $(\mathcal{B}, F)$ is called a *computable Banach space*.
- Since $\| \ \|$ is computable on $\mathcal{L}_{\mathbb{Q}(i)}(F)$, this gives us a computable metric space. So concepts like 'computable vector (with respect to F)', 'computable operator with respect to (F_1, F_2)' etc. are now defined.
- $+$ and $\cdot$ are computable with respect to F.
- Key issue: when working with an effective generating set F, we can 'see' the norms of vectors in $\mathcal{L}_{\mathbb{Q}(i)}(F)$ but we can not see their components. i.e. the functional $e_n^*(a_0, a_1, \ldots) = a_n$ may not be computable with respect to F.

Some simple observations

Suppose p is a computable real so that $p \geq 1$.

- A vector is computable with respect to E if and only if it is a computable sequence of complex numbers.
- ζe_0 is computable with respect to ζE (even if ζ is incomputable!)
- The map $T : \ell^p \to \ell^p$ given by $T(f) = \zeta f$ is computable with respect to $(E, \zeta E)$.
- So ζE 'yields same computability theory on ℓ^p' as E.

Fundamental definition

Recall that a norm-preserving map is called an *isometry*.

Definition 2 (A.G. Melnikov 2013)

Suppose $\mathcal{B}$ is a computable Banach space. $\mathcal{B}$ is *computably categorical* if it is the case that whenever $\mathcal{B}_0$ is a computable Banach space that is linearly isometric to $\mathcal{B}$, there is a computable linear isometry of $\mathcal{B}$ onto $\mathcal{B}_0$.

Thus, $\mathcal{B}$ is computably categorical if and only if it is the case that whenever F_0 and F_1 are effective generating sets for $\mathcal{B}$, there is an isometric endomorphism of $\mathcal{B}$ that is computable with respect to (F_1, F_2).

Prior results

- Pour-El and Richards 1989: ℓ^1 is not computably categorical.
- Melnikov 2013: ℓ^2 is computably categorical.
- M. 2015: ℓ^p computably categorical iff $p = 2$.
- M. 2015: ℓ^p Δ^0_2 categorical

The third result is a consequence of the following.

Theorem 3 (M. 2015)

Suppose p is a computable real so that $p \geq 1$ and $p \neq 2$. Suppose C is a c.e. set. Then, there is a computable copy of ℓ^p, $\mathcal{B}$, so that C computes a linear isometry of ℓ^p onto $\mathcal{B}$ and so that any oracle that computes a linear isometry of ℓ^p onto $\mathcal{B}$ must also compute C.

The proof of this theorem is based on the following classification of the isometries of ℓ^p.

Theorem (Banach/Lamperti)

Suppose $1 \leq p < \infty$ and $p \neq 2$. Let T be a linear map of ℓ^p into ℓ^p. Then, the following are equivalent.

1. *T is a surjective isometry.*
2. *There is a permutation of $\mathbb{N}$, ϕ, and a sequence of unimodular points, $\{\lambda_n\}_n$, so that $T(e_n) = \lambda_n e_{\phi(n)}$ for all n.*

This classification is an immediate consequence of the following.

Theorem (Banach/Lamperti)

Suppose $1 \leq p < \infty$ *and* $p \neq 2$. *If* $T : \ell^p \to \ell^p$ *is a linear isometry, then* T *preserves the 'disjoint support' relation. i.e. for all* $f, g \in \ell^p$

$$\operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset \;\Rightarrow\; \operatorname{supp}(T(f)) \cap \operatorname{supp}(T(g)) = \emptyset.$$

Definition 4

Suppose p is a computable real so that $p \geq 1$. Suppose $\mathcal{B}$ is a computable Banach space that is is linearly isometric to ℓ^p. The *isometry degree* of $\mathcal{B}$ is the least powerful oracle that computes a linear isometry of ℓ^p onto $\mathcal{B}$.

That is, **C** is the isometry degree of $\mathcal{B}$ if **C** computes a linear isometry of ℓ^p onto $\mathcal{B}$ and if **C** $\leq$ **D** whenever **D** computes a linear isometry of ℓ^p onto $\mathcal{B}$.

Observations: Suppose p is a computable real so that $p \geq 1$.

- When $p = 2$, $\mathbf{0}$ is the only degree of linear isometry.
- When $p \neq 2$ every c.e. degree is a degree of linear isometry for a computable copy of ℓ^p.

Questions:

- Does every computable copy of ℓ^p have a degree of isometry?
- Is every such degree a c.e. degree?

Theorem 5 (M., Stull 2016)

Suppose p is a computable real so that $p \geq 1$. Every computable copy of ℓ^p has a degree of linear isometry and this degree is a c.e. degree

Corollary 6 (M., Stull 2016)

Suppose p is a computable real so that $p \geq 1$ and $p \neq 2$. Then, the isometry degrees of computable copies of ℓ^p are precisely the c.e. degrees.

A new (and better) proof of Theorem 3 (due to M., Stull)

Preliminaries from ℓ^p spaces: Suppose $1 \leq p < \infty$ and $p \neq 2$.

- When $f, g \in \ell^p$, write $f \preceq g$ if

$$f(n) \neq g(n) \Rightarrow f(n) = 0.$$

- It follows that $\preceq$ is a partial order. The atoms are the vectors of the form λe_n where $\lambda \neq 0$.
- $f \preceq g$ iff f and $g - f$ are disjointly supported.
- Thus, $\preceq$ is preserved by linear isometries; atoms are mapped to atoms.

New proof of Theorem 3

Construction of $\mathcal{B}$ and F: Now, suppose p is computable.

- Let C be an infinite c.e. set; let $\{C_s\}_s$ be a computable enumeration of C.
- Start by putting $e_{2n} + e_{2n+1}$ into F for each $n \in \mathbb{N}$.
- If n later enters C, put e_{2n} into F.
- Let $\mathcal{B} =$ subspace of ℓ^p generated by F.

New proof of Theorem 3

Sketch of verification:

- $e_{2n} + e_{2n+1}$ is an atom of $(\mathcal{B}, \preceq)$ if and only if $n \notin C$; otherwise e_{2n} and e_{2n+1} are atoms.
- $\mathcal{B}$ is spanned by a countably infinite set of disjointly supported vectors. Thus, $\mathcal{B}$ is linearly isometric to ℓ^p.
- If X computes an isometry of ℓ^p onto $\mathcal{B}$, then can use X to determine which vectors $e_{2n} + e_{2n+1}$ "split" during formation of F.
- C computes a linear isometry of ℓ^p onto $\mathcal{B}$.

Proof degrees isometry = c.e. degrees

Suppose p is a real so that $p \geq 1$ and $p \neq 2$.

- We consider maps $\phi :\subseteq \omega^{<\omega} \to \ell^p$ that are order-preserving in the sense that

$$\sigma \subset \tau \;\Rightarrow\; \phi(\tau) \prec \phi(\sigma).$$

- Such a map is *strongly order-preserving* if it maps incomparable nodes to disjointly supported vectors.

Theorem 7 (M. 2015)

Suppose F is an effective generating set for ℓ^p. Then, there is a c.e. subtree T of $\omega^{<\omega}$ and a strongly order-preserving map $\phi : T \to \ell^p$ with the following two features.

1. *If $\phi(\sigma)$ is not an atom, then σ is not a leaf node of T and*

$$\phi(\sigma) = \sum_{\sigma^\frown(j) \in T} \phi(\sigma^\frown(j)).$$

2. $\mathrm{ran}(\phi)$ *is a generating set for ℓ^p.*

Throughout rest of talk, F, T, ϕ are are in Theorem 7.

Definition 8

A chain $C \subseteq T$ is *almost norm-maximizing* if for every non-leaf $\sigma \in C$ there exists j so that $\sigma^\frown(j) \in C$ and so that

$$\max\{\|\phi(\sigma^\frown(k))\|_p^p : \sigma^\frown(k) \in T\} \leq \|\phi(\sigma^\frown(j))\|_p^p + 2^{-|\sigma|}$$

Proposition 1 (M. 2015)

If C is an almost norm-maximizing chain, then $\inf \phi[C]$ *is either* $\mathbf{0}$ *or an atom.*

Theorem 9 (M. 2015)

There is a computable partition of T into almost norm-maximizing chains.

The following theorem allows us to say what the degree of isometry is for a computable copy of ℓ^p.

Theorem 10 (M., Stull 2016)

If $\{C_0, C_1, \ldots\}$ is a partition of T into almost norm-maximizing chains, and if $g_n = \inf \phi[C_n]$ for all n, then the Turing degree of $\{\|g_n\|_p\}_n$ is the degree of linear isometry of (ℓ^p, F) and is a c.e. degree.

Proof Sketch

- When $f, g \in \ell^p$, set

$$\sigma(f,g) = |\|f+g\|_p^p + \|f-g\|_p^p - 2\|f\|_p^p - 2\|g\|_p^p|$$

- By a Theorem of Lamperti: f, g disjointly supported if and only if $\sigma(f,g) = 0$.
- Set $\kappa_n = |C_n|$
- When $j < \kappa_n$, set $\sigma_{n,j} = (j+1)st$-element C_n in $\subset$-ascending order.
- Set $g_{n,j} = \phi(\sigma_{n,j})$
- So, $g_n = \lim_j g_{n,j}$.

Suppose oracle X computes isometric endomorphism S of ℓ^p with respect to (E, F). Given $k \in \mathbb{N}$, to compute $\|g_n\|_p$ with precision 2^{-k} wait until one of following happens:

- We see that $\|g_n\|_p < 2^{-k}$.
- We find m, j so that $\text{supp}(S(e_m)) \cap \text{supp}(g_{n,j}) \neq \emptyset$ and

$$\|g_{n,j} - S(e_m)^*(g_{n,j})S(e_m)\|_p < 2^{-(k+1)}$$

and

$$\|g_{n,j} - S(e_m)^*(g_{n,j})S(e_m)\|_p^p + 2^{-|\sigma_{n,j}|} < \|S(e_m)^*(g_{n,j})S(e_m)\|_p^p.$$

In the first case, output 0. In the second case, compute a rational number q so that $|q - \|g_{n,j}\|_p| < 2^{-(k+1)}$. Output q

Summary

Suppose p is a computable real so that $p \geq 1$ and $p \neq 2$.

- If $\mathcal{B}$ is a computable copy of ℓ^p, the isometry degree of $\mathcal{B}$ is the least powerful degree that computes an isometry of ℓ^p onto $\mathcal{B}$.
- Every computable copy of ℓ^p has an isometry degree.
- The isometry degrees (for computable copies of ℓ^p) are precisely the c.e. degrees.