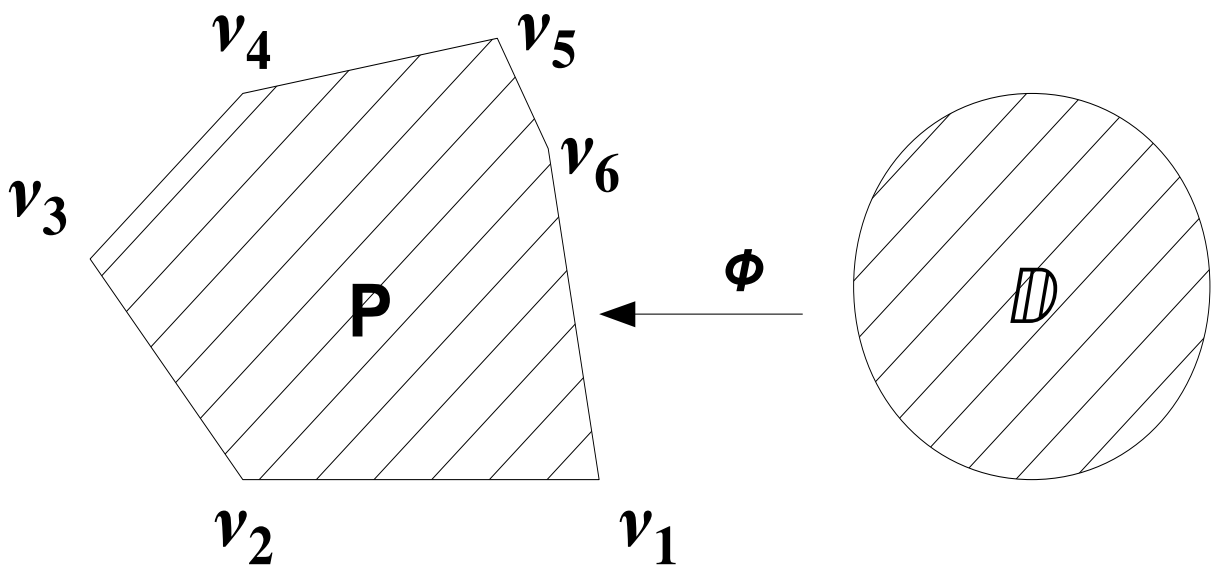
A potential-theoretic construction of the Schwarz-Christoffel map for multiply connected domains

Special Session on Computational and Applied Mathematics
AMS Fall Central Section Meeting, 2011

Timothy H. McNicholl (with Valentin V. Andreev)

timothy.h.mcnicholl@gmail.com
Department of Mathematics
Lamar University
Beaumont, Texas 77710 USA

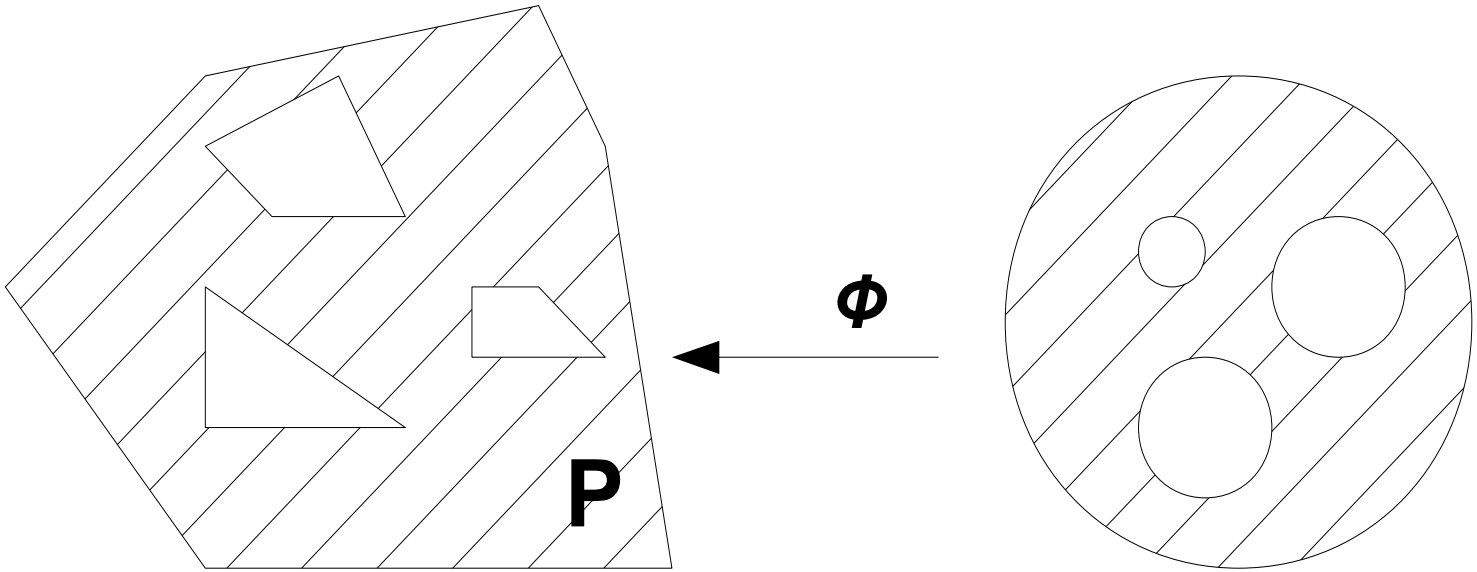
October, 2011

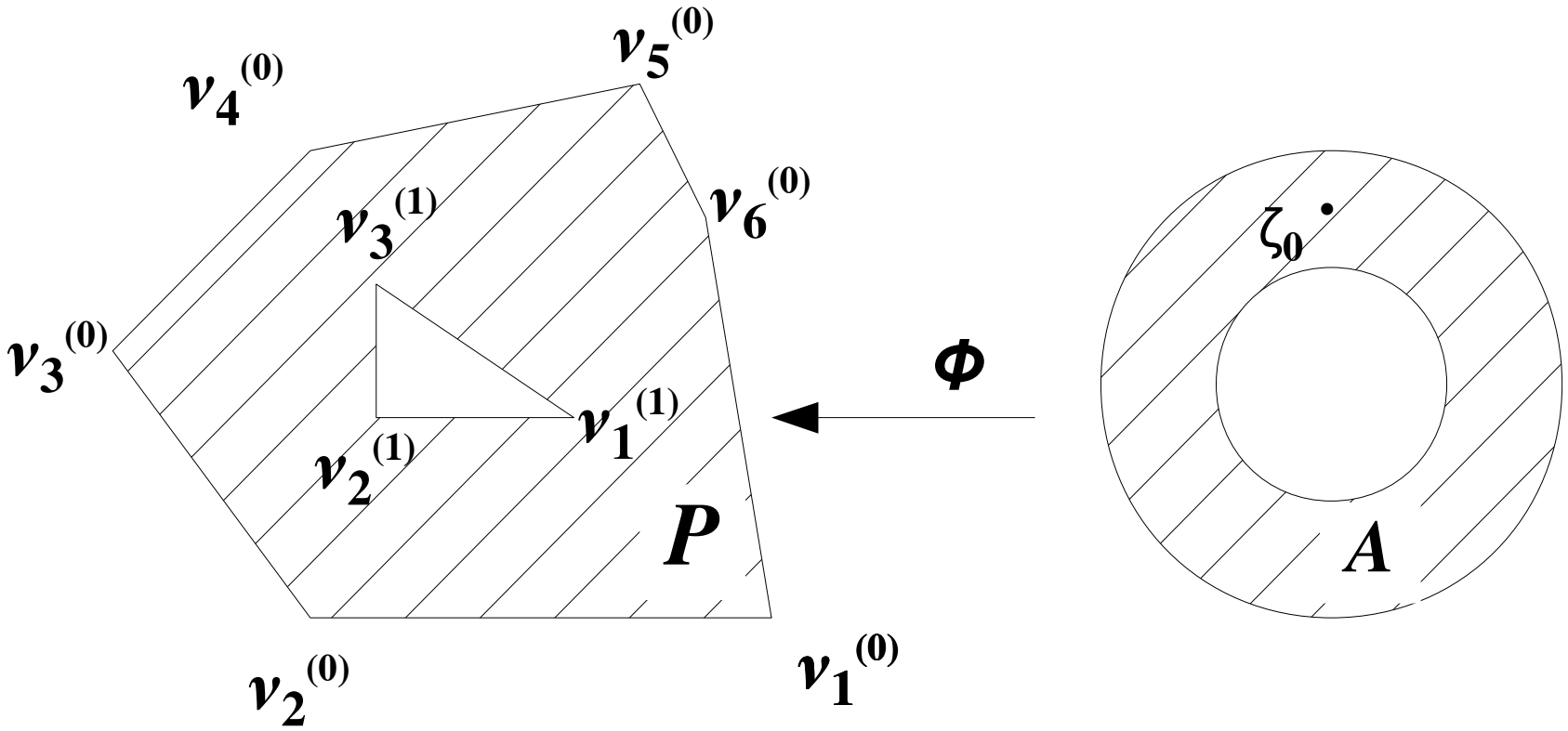

ν_4
ν_5
ν_6
ν_3
P
Φ
D
ν_2
ν_1

Schwarz-Christoffel formula

$$
\begin{aligned}
n &= \text{number of vertices} \\
a_j &= \phi^{-1}(\nu_j)\ (\textit{pre-vertices}) \\
\pi(1-\beta_j) &= \text{turning angle at } \nu_j,\ 0 < \beta_j < 1
\end{aligned}
$$

$$
\phi(z) = A \int_0^z \prod_{k=1}^{n_0} (\zeta - a_k)^{\beta_k - 1} d\zeta + B
$$


P
ϕ


$\nu_4^{(0)}$
$\nu_5^{(0)}$
$\nu_3^{(1)}$
$\nu_6^{(0)}$
$\nu_3^{(0)}$
$\nu_1^{(1)}$
$\nu_2^{(1)}$
P
$\nu_2^{(0)}$
$\nu_1^{(0)}$
Φ
ζ_0
A

$$
\begin{aligned}
0 &= \text{center of } A \\
1 &= \text{outer radius } A \\
\mu &= \text{inner radius } A \\
a_1^{(0)} \dots a_{n_0}^{(0)} &= \text{pre-vertices on } \partial\mathbb{D} \\
a_1^{(1)} \dots a_{n_1}^{(1)} &= \text{pre-vertices on } \partial D_\mu(0) \\
\pi(1-\beta_k^{(j)}) &= \text{turning angle at } \nu_k^{(j)},\ 0 < \beta_k^{(j)} < 1 \\
\alpha_{1,k} &= 2 - \beta_k^{(1)} \\
\alpha_{0,k} &= \beta_k^{(0)}
\end{aligned}
$$

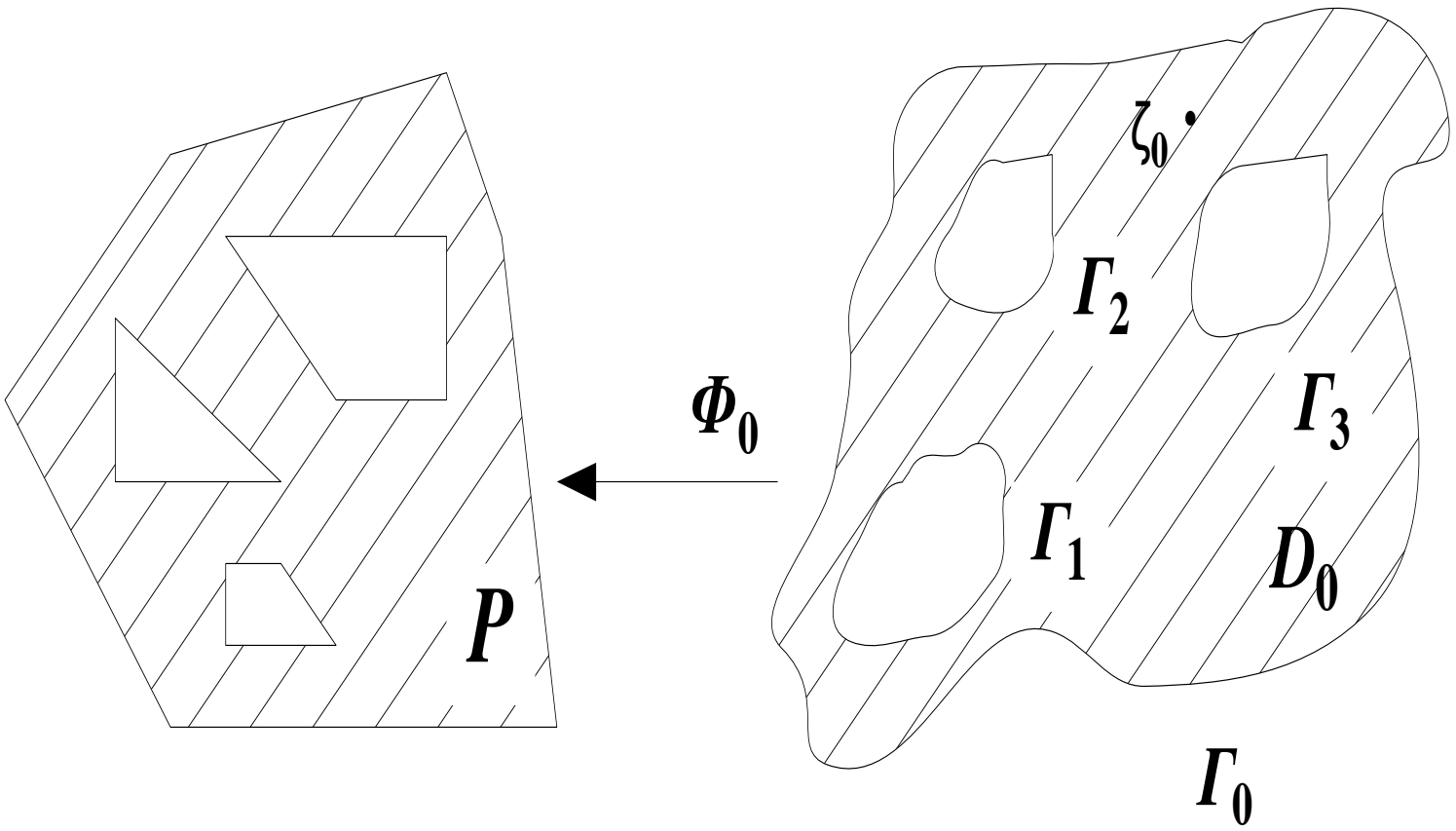
$$\phi(z) = B\int_{\zeta_0}^{z}\prod_{m=1}^{n_0}\left(\Theta\left(\frac{\zeta}{\mu a_m^{(0)}}\right)\right)^{\alpha_{0,m}-1}\prod_{m=1}^{n_1}\left(\Theta\left(\frac{\mu\zeta}{a_m^{(1)}}\right)\right)^{\alpha_{1,m}-1}d\zeta + C$$

$$\Theta(z) = \prod_{k=1}^{\infty}(1-\mu^{2k-1}z)(1-\mu^{2k-1}z^{-1}).$$

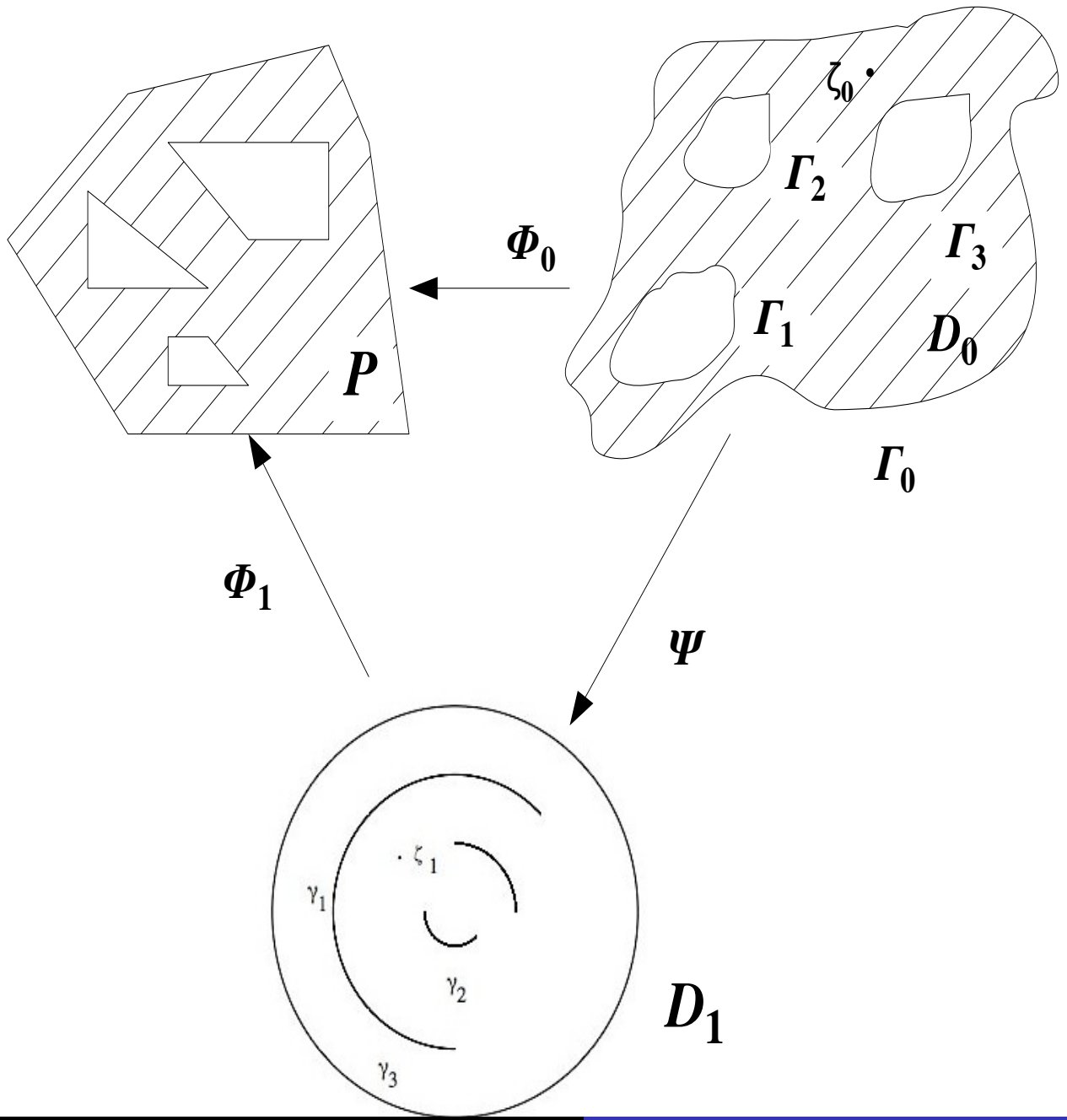
- L.E. Dunduučenko and S.V. Gončarenko, 1978.
- DeLillo, 2006 (and also DeLillo, Elcrat, Pfaltzgraf 2004).
- Crowdy, 2005

Central claims

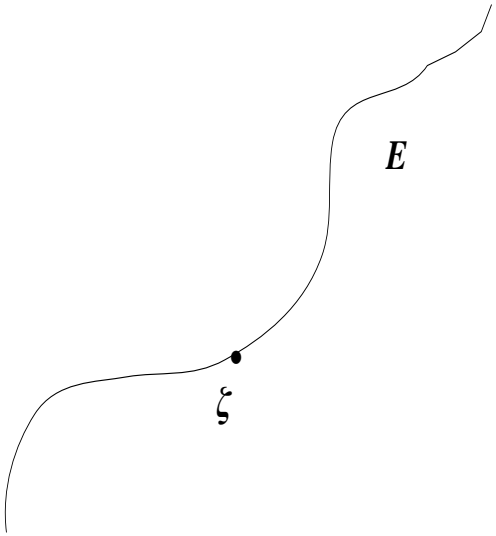
- The Schwarz-Christoffel map *for Jordan domains* can be expressed in terms of the functions of potential theory at all levels of connectivity. (*i.e.* a representation in the spirit of Max Schiffer.)
- One formula works for all levels of connectivity and for mapping to bounded or unbounded polygonal domains.


ζ_0
Γ_2
Γ_3
Φ_0
Γ_1
D_0
P
Γ_0

$$
\begin{aligned}
\Gamma_0, \ldots, \Gamma_M &= \text{boundary curves of } D_0,\ \Gamma_0 \text{ outermost} \\
a_1^{(j)}, \ldots, a_{n_j}^{(j)} &= \text{pre-vertices on } \Gamma_j \\
\pi(1 - \beta_k^{(j)}) &= \text{turning angle at } \phi_0(a_k^{(j)}),\ 0 < \beta_k^{(j)} < 1
\end{aligned}
$$

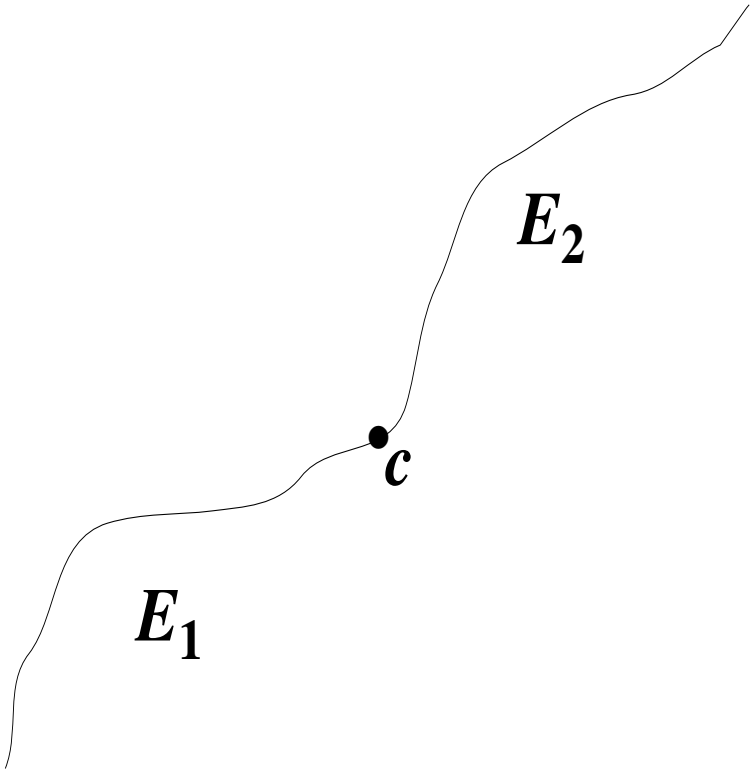

ζ_0
Γ_2
Γ_3
Φ_0
Γ_1
D_0
P
Γ_0
Φ_1
ψ
ζ_1
γ_1
γ_2
γ_3
D_1

$S =_{df} \partial D_0 - \{\text{pre-vertices}, \text{pre-endpoints}\}$



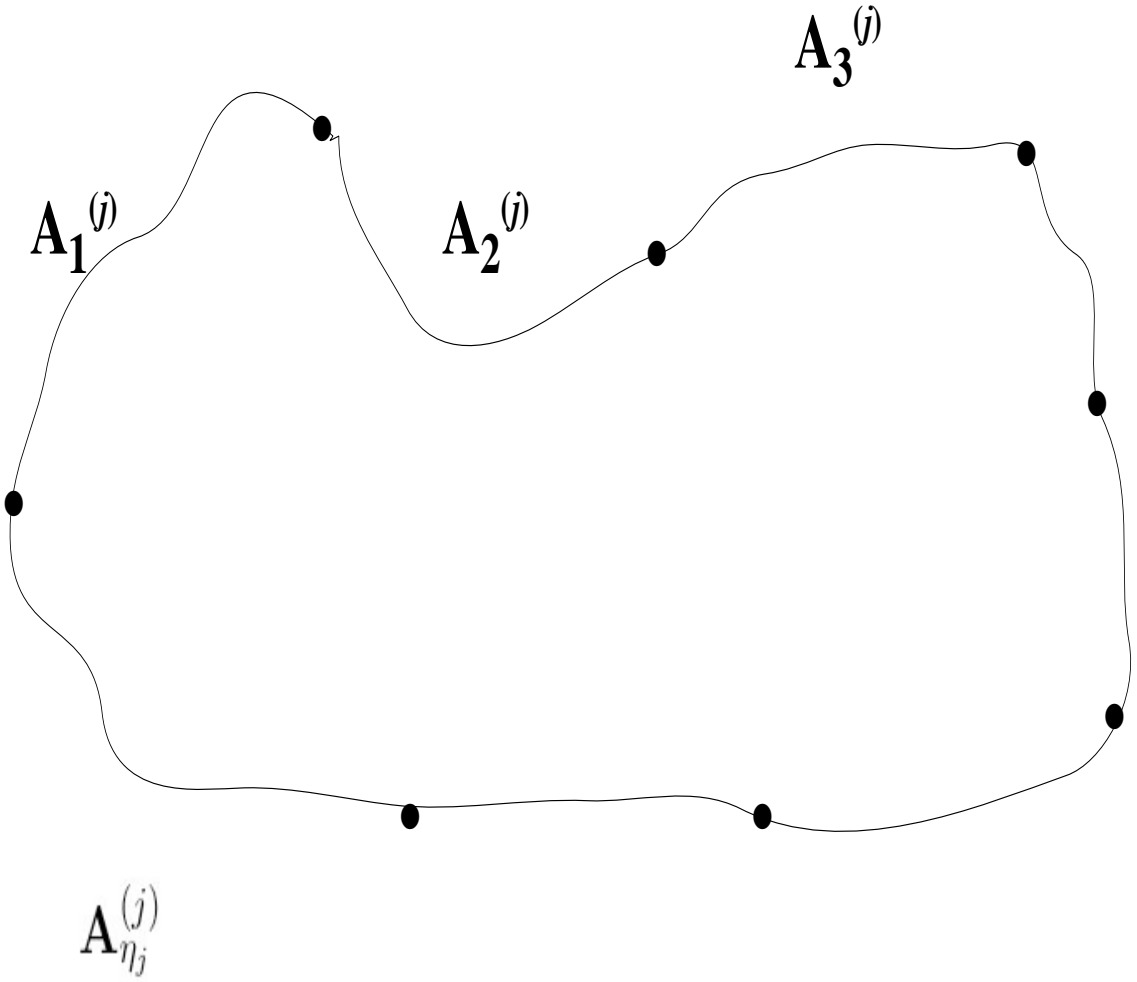
$E \subseteq S$ connected

On E, $\text{Arg}(\psi(\zeta)\phi_1'(\psi(\zeta))) = \text{a constant}, C_E$



E_1, E_2 connected components of S.

$$C_{E_2} - C_{E_1} \equiv \begin{cases} -\pi & c \in \{\text{pre-ep.'s}\} - \{\text{pre-vertexes}\} \\ \pi(1 - \beta_k^{(j)}) & c = a_k^{(j)} \ \& \ c \notin \{\text{pre-ep.'s}\} \\ -\pi\beta_k^{(j)} & c = a_k^{(j)} \ \& \ c \in \{\text{pre-ep.'s}\} \end{cases}$$



$$A_{\eta_{j+1}}^{(j)} = A_1^{(j)}, A_0^{(j)} = A_{\eta_j}^{(j)}$$

$$
\begin{aligned}
\theta_k^{(j)} &= C_{A_k^{(j)}} - C_{A_{k-1}^{(j)}} \\
\sigma_1^{(j)} &= 0 \\
\sigma_2^{(j)} &= \theta_2^{(j)} \\
\sigma_k^{(j)} &= \sigma_{k-1}^{(j)} + \theta_k^{(j)} \;\; k \geq 3
\end{aligned}
$$

$v_{0,1}$ harmonic on D_0,

$$v_{0,1}(\zeta) = \sigma_k^{(j)} - \mathrm{Arg}(\psi(\zeta)) \text{ if } \zeta \in A_k^{(j)}.$$

$$v_{0,1} - \mathrm{Arg}(\phi_1' \circ \psi) \text{ constant on each } \Gamma_j$$

$\omega_j =_{df}$ harmonic measure of Γ_j.

Theorem

If u is harmonic on D_0, then there exist unique real $b_1(u), \ldots, b_M(u)$ such that

$$u + \sum_j b_j(u)\omega_j$$

has a harmonic conjugate.

$$v_{0,2} \quad =_{df} \quad v_{0,1} + \sum_j b(v_{0,1})\omega_j$$

$$u_{0,2}(z) \quad =_{df} \quad -\int_{\zeta_0}^{z} \frac{\partial v_{0,2}}{\partial \eta} ds$$

$v_{0,2} - \mathrm{Arg}(\phi_1' \circ \psi)$ constant on each Γ_j

$u_{0,2} + iv_{0,2} \quad \text{“}=\text{”} \quad \mathrm{Log}(\phi_1' \circ \psi)$

$$
\begin{aligned}
u_{1,2} &=_{df} u_{0,2} \circ \psi^{-1} \\
v_{1,2} &=_{df} u_{1,2} \circ \psi^{-1} \\
H &=_{df} \exp(u_{1,2} + i v_{1,2}) (\text{“} = \text{”} \phi_1') \\
\zeta_1 &=_{df} \psi(\zeta_0) \\
f_1(z) &=_{df} \int_{\zeta_0}^{z} H(\zeta) d\zeta \\
f_0 &=_{df} f_1 \circ \psi
\end{aligned}
$$

Verification sketch:

- $\mathrm{Arg}(f_0'/\phi_0')$ is constant.
- f_0'/ϕ_0' is constant.
- $\phi_0 = af_0 + b$.

$$
\begin{aligned}
P_{k,j} &=_{df} \int_{\Gamma_k} \frac{\partial \omega_j}{\partial \eta} ds \\
P &=_{df} \left(P_{k,j}\right)_{k,j} \\
p_j(u) &=_{df} \int_{\Gamma_j} \frac{\partial u}{\partial \eta} ds
\end{aligned}
$$

$$
\begin{pmatrix} b_1(u) \\ . \\ . \\ . \\ b_M(u) \end{pmatrix} = P^{-1} \begin{pmatrix} p_1(u) \\ . \\ . \\ . \\ p_M(u) \end{pmatrix}
$$

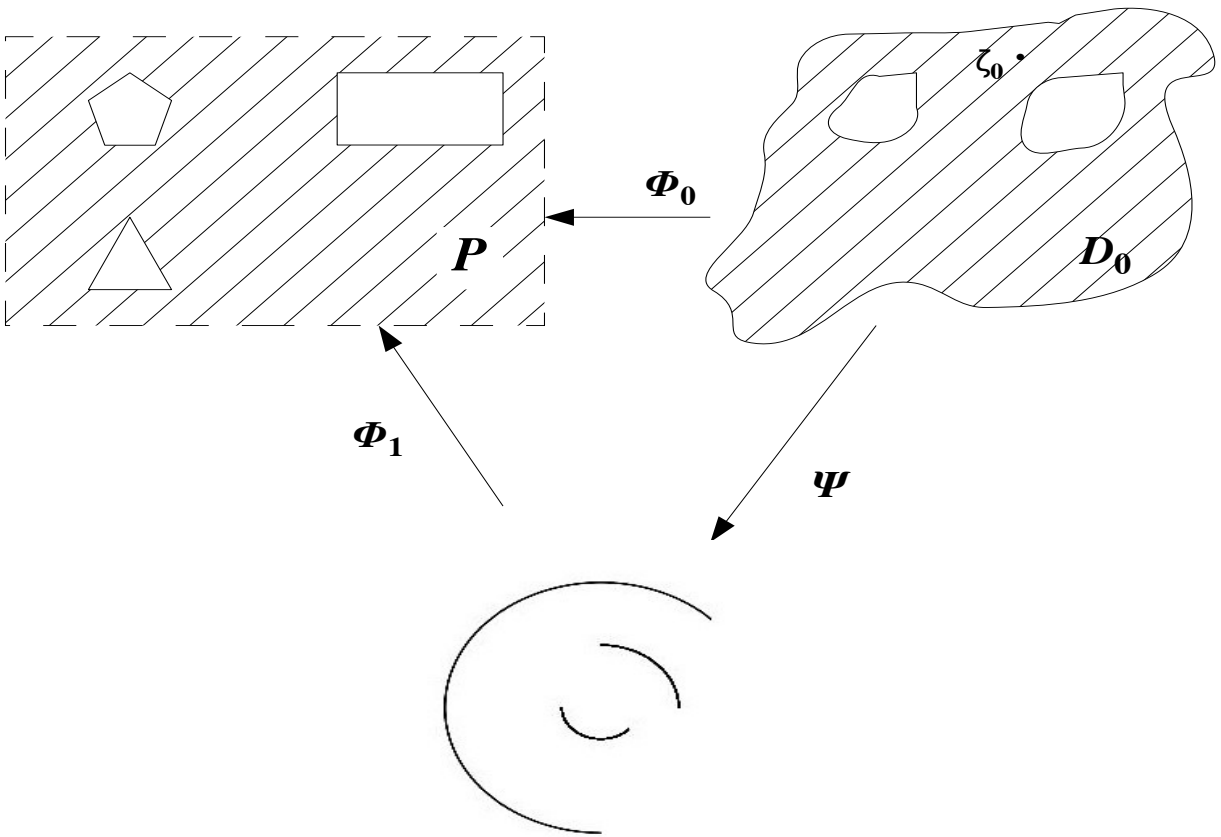
$$\begin{aligned} \mathrm{Re}(\hat{u}) &= u + \sum_j b_j(u)\omega_j \\ G(z) &=_{df} G_{D_0}(z, \zeta_0) \end{aligned}$$

$$\psi(z) = \exp(-\hat{G}(z))$$

$$\Lambda(z) =_{df} \sum_{k,j} \sigma_k^{(j)} \omega(z, A_k^{(j)}, D_0)$$

$$\Omega(z) =_{df} -\frac{1}{2\pi} \int_{\partial D_0} \mathrm{Arg}(\psi(\zeta)) \frac{\partial G_{D_0}(z,\zeta)}{\partial \eta_\zeta} ds_\zeta$$

$$\begin{aligned} v_{0,2} &= \Lambda - \Omega \\ u_{0,2} + i v_{0,2} &= (-i)(\hat{\Lambda} - \hat{\Omega}) \\ f_0(z) &= \int_{\zeta_0}^{z} \exp\left\{(-i)(\hat{\Lambda}(\zeta) - \hat{\Omega}(\zeta))\right\} d\zeta \end{aligned}$$


ζ_0
Φ_0
P
D_0
Φ_1
Ψ

Derivation of the disk formula

WLOG assume $a_1^{(0)} = -1$. n.b. $\psi = Id$.

$$\begin{aligned}
\hat{\Lambda}(z) &= \frac{1}{\pi}\left(\sum_{k=1}^{n_0}(-\theta_k^{(0)})\,\mathrm{Log}(z - a_k^{(0)}) + 2\pi\,\mathrm{Log}(1+z)\right) + C \\
\hat{\Omega}(z) &= 2\,\mathrm{Log}_1(1+z) + \mathit{Const.}
\end{aligned}$$

WLOG assume $\zeta_0 = \tau \in \mathbb{R}$. Let $b_0, b_1 =$ endpoints of γ_1.

$$\exp(\hat{\Lambda}(z)) =$$

$$\frac{\Theta(\mu^{-1}z/a_1^{(0)})^2}{\Theta(\mu^{-1}z/b_0)\Theta(\mu^{-1}z/b_1)}$$

$$\times \prod_{m=1}^{n_0} \Theta\left(\mu^{-1}\frac{z}{a_m^{(0)}}\right)^{\beta_m^{(0)}-1} \prod_{m=1}^{n_1} \Theta\left(\mu^{-1}\frac{z}{a_m^{(1)}}\right)^{1-\beta_m^{(1)}}$$

$$\exp(\hat{\Omega}(z)) = \Theta(\mu^{-1}z/a_1^{(0)})^2 \left(1 - \frac{z}{\tau}\right) \Theta(\mu^{-1}z/\tau)^{-1}\Theta(\mu z/\tau)^{-1}$$

$$N(z) =_{df} \frac{\Theta(\mu^{-1}z/\tau)\Theta(\mu z/\tau)\hat{G}'(z)}{\Theta(\mu^{-1}z/b_0^{(1)})\Theta(\mu^{-1}z/b_1^{(1)})}$$

$$f_0'(z) \quad = \quad N(z)\prod_{m=1}^{n_0}\Theta\left(\mu^{-1}\frac{z}{a_m^{(0)}}\right)^{\beta_m^{(0)}-1}\prod_{m=1}^{n_1}\Theta\left(\mu^{-1}\frac{z}{a_m^{(1)}}\right)^{1-\beta_m^{(1)}}$$

Loxodromic function: $f(\mu^2 z) = f(z)$.

$$N(\mu^2 z) = \mu^{-4} N(z)$$

$z \mapsto z^2 N(z)$ loxodromic and analytic except at 0

$z \mapsto z^2 N(z)$ constant (Liouiville's Theorem)

$$
\begin{aligned}
f_0'(z) &= Cz^{-2} \prod_{m=1}^{n_0} \Theta\left(\mu^{-1} \frac{z}{a_m^{(0)}}\right)^{\beta_m^{(0)}-1} \prod_{m=1}^{n_1} \Theta\left(\mu^{-1} \frac{z}{a_m^{(1)}}\right)^{1-\beta_m^{(1)}} \\
&= C\phi_0'(z).
\end{aligned}
$$

Conclusions

- Unified approach to Schwarz-Christoffel maps.
- Possible convergent formulas for circular domains in terms of rational functions.
- Possible new methods of computation.

,


V.V. Andreev and T.H. McNicholl, *A potential-theoretic construction of the Schwarz-Christoffel map for finitely connected domains*, To appear in Complex Variables and Elliptic Equations. Available online at www.tandfonline.com. DOI: 10.1080/17476933.2011.561333.


D. Crowdy, *The Schwarz-Christoffel mapping to bounded multiply connected polygonal domains*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **461** (2005), no. 2061, 2653–2678.


T. K. DeLillo, A. R. Elcrat, and J. A. Pfaltzgraff, *Schwarz-Christoffel mapping of multiply connected domains*, J. Anal. Math. **94** (2004), 17–47.


Thomas K. DeLillo, *Schwarz-Christoffel mapping of bounded, multiply connected domains*, Comput. Methods Funct. Theory **6** (2006), no. 2, 275–300.

L. E. Dundučenko and S. V. Gončarenko, *The Christoffel-Schwarz formula generalized on n-connected circular domains*, Vyčisl. Prikl. Mat. (Kiev) (1978), no. 35.

P. Henrici, *Applied and computational complex analysis. Vol. 3*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1986, Discrete Fourier analysis—Cauchy integrals—construction of conformal maps—univalent functions, A Wiley-Interscience Publication.