Index sets of classes of Lebesgue spaces

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A little background/history

- Index sets have been studied for a long time in computability theory and computable algebra.
- The usual problem is to determine the complexity within the hyperarithmetical hierarchy of the index set associated with a classification problem.
- Recently, a number of results have been obtained on index sets associated with problems in analysis. These findings concern various classes of metric spaces.
- Today I will discuss some recent work on index sets associated with classes of Banach spaces. Let's start with a quick review of Banach spaces.

Banach space basics

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $\mathcal{V} = (V, \cdot, +, \| \ \|_{\mathcal{V}})$ be a normed vector space over $\mathbb{F}$.

- $\mathcal{V}$ is a metric space with $d(u, v) = \|u - v\|_{\mathcal{V}}$.
- If this metric space is complete, then $\mathcal{V}$ is a *Banach space*.
- Recall that a *Hilbert space* is a Banach space where the norm is induced by an inner product.

Example

$\mathbb{F}^n$ is a Banach space under the 'usual' norm:

$$\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \ldots + x_n^2}.$$

This is also a Hilbert space.

More interesting examples forthcoming.

Presentations of Banach spaces

Definition

Let $\mathcal{B}$ be a Banach space.

1. $R : \mathbb{N} \to \mathcal{B}$ is a *structure* on $\mathcal{B}$ if ran(R) is linearly dense (i.e. its linear span is dense in $\mathcal{B}$.
2. If R is a structure on $\mathcal{B}$, then $(\mathcal{B}, R)$ is a *presentation* of $\mathcal{B}$.

Convention

Presentations of a Banach space $\mathcal{B}$ are denoted $\mathcal{B}^\#$ or $\mathcal{B}^+$.

Computable presentations of Banach spaces

Definition

Suppose $\mathcal{B}^\# = (\mathcal{B}, R)$ is a presentation of $\mathcal{B}$. $v \in \mathcal{B}$ is a *rational vector* of $\mathcal{B}^\#$ if there exist rational scalars $\alpha_0, \ldots, \alpha_M$ so that $v = \sum_{j=0}^{M} \alpha_j R(j)$.

Definition

Suppose $\mathcal{B}^\#$ is a presentation of a Banach space $\mathcal{B}$. $\mathcal{B}^\#$ is a *computable presentation* if there is an algorithm that given a (code of) a rational vector v and an integer k produces a rational number q so that $|q - \|v\|_{\mathcal{B}}| < 2^{-k}$; every index of such an algorithm is an *index* of $\mathcal{B}^\#$.

Remark

If e indexes a presentation of $\mathcal{B}$, then e indexes a presentation of every space isometrically isomorphic to $\mathcal{B}$.

Two typical kinds of problems

Problem

For various natural classes $\mathcal{K}$ of Banach spaces, determine the complexity of

$$\{e \mid e \text{ indexes a computable presentation of a } \mathcal{B} \in \mathcal{K}\}.$$

This corresponds to complexity of classifying spaces in $\mathcal{K}$.

Problem

For various natural classes $\mathcal{K}$ of Banach spaces, determine the complexity of

$$\{(e, e') \mid e, e' \text{ index computable presentations of a } \mathcal{B} \in \mathcal{K}\}.$$

We start by examining Lebesgue spaces; let's review what these are.

Review of Lebesgue spaces

Motivation

Recall from calculus $\int_0^1 \frac{1}{x^p} dx < \infty$ *if and only if* $p < 1$. L^p *spaces are the ultimate generalization of this sort of problem.*

Definition

Let $\Omega = (X, \mathcal{S}, \mu)$ be a measure space, and suppose $1 \leq p < \infty$. Then, $L^p(\Omega) =$ set of all measurable $f : X \to \mathbb{F}$ so that $\int |f|^p \, d\mu < \infty$.

Review of Lebesgue spaces

Remark

Assume $1 \leq p < \infty$*.*

1. $L^p(\Omega)$ *is a vector space under pointwise addition and scalar multiplication.*
2. *We identify functions in* $L^p(\Omega)$ *that agree a.e..*
3. *If* $p < \infty$*, then* $L^p(\Omega)$ *is a Banach space under the norm*

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{1/p}.$$

Review of Lebesgue spaces

Definition

1. An L^p *space* is a Banach space of the form $L^p(\Omega)$.
2. A Banach space is a *Lebesgue* space if it is an L^p space for some $p \geq 1$.

Example

1. $L^p[0,1] = L^p([0,1], \mathcal{S}, m)$ where $\mathcal{S}$ is the σ-algebra of Lebesgue measurable subsets of $[0,1]$ and $m =$ Lebesgue measure on $[0,1]$.
2. $\ell^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where μ is the counting measure on $\mathbb{N}$. (so $\|\{s_n\}_{n=0}^\infty\|_p = \left(\sum_{n=0}^\infty |s_n|^p\right)^{1/p}$).
3. $\ell^p_n = L^p(\{1, \ldots, n\}, \mathcal{P}(\mathbb{N}), \mu)$ where μ is the counting measure on $\{1, \ldots, n\}$.

L^p spaces are common objects of study in various branches of analysis.

Indices of L^p-space presentations (fixed p)

We decided to first focus on the class of L^p spaces for fixed computable $p \geq 1$. The associated classification problem is “Which Banach spaces are isometrically isomorphic to an L^p space?"

Theorem (Kakutani Representation Theorem (1941))

Suppose $1 \leq p < \infty$. Let $\mathcal{B}$ be a Banach space. Then, $\mathcal{B}$ is isometrically isomorphic to an L^p space if and only if there is a lattice ordering $\leq$ of $\mathcal{B}$ so that:

1. $x \leq y \Rightarrow x + z \leq y + z$
2. $x \leq y \ \& \ t \geq 0 \Rightarrow tx \leq ty$
3. $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$ *(where $|x| = x \vee (-x)$).*
4. $\|x + y\|^p = \|x\|^p + \|y\|^p$ *whenever* $|x| \wedge |y| = 0$.

Indices of L^p-space presentations (fixed p)

Theorem (Brown, M., Melnikov 2018+)

Suppose $p \geq 1$ is a computable real. Then, the set of all indices of computable presentations of L^p spaces is Π^0_2-complete.

The low complexity is likely due in part to the fact that we are only looking at separable L^p spaces, and these have another classification.

Classification of separable L^p spaces

Theorem

Every L^2 space is a Hilbert space. Every separable L^2 space is a isometrically isomorphic to ℓ^2 or to ℓ^2_n for some n.

Theorem

Suppose $1 \leq p < \infty$ and $p \neq 2$. Every nonzero separable L^p space is isometrically isomorphic to exactly one of the following.

1. ℓ^p_n
2. ℓ^p
3. $L^p[0,1]$
4. *An external direct sum of two of the above with L^p-norm. (i.e. $\|(f,g)\|_p^p = \|f\|_p^p + \|g\|_p^p$.)*

Results for subclasses of L^2

Theorem (Brown, M., Melnikov 2018+)

1. *The set of all indices of computable presentations of* ℓ^2 *is* Π^0_2*-complete.*
2. *If* $n > 1$*, then the set of all indices of computable presentations of* ℓ^2_n *is* Π^0_2*-complete.*

Results for subclasses of L^p, $p \neq 2$

Theorem (Brown, M., Melnikov 2018+)

Let p be a computable real so that $p \geq 1$ and $p \neq 2$.

1. *For each $n \geq 1$, the set of all indices of computable presentations of ℓ^p_n is Π^0_2-complete.*
2. *The set of all indices of computable presentations of ℓ^p is Π^0_3-complete.*
3. *The set of all indices of computable presentations of $L^p[0,1]$ is Π^0_2-complete.*
4. *For each $n \geq 1$, the set of all indices of computable presentations of $\ell^p_n \oplus_p L^p[0,1]$ is d-Σ^0_2-complete.*
5. *The set of all indices of computable presentations of $\ell^p \oplus_p L^p[0,1]$ is d-Σ^0_3-complete.*

The isometric isomorphism problem

Theorem (Brown, M., Melnikov 2018+)

Let p be a computable real so that $p \geq 1$ and so that $p \neq 2$. Then, the set of all (e, e') so that e, e' index computable presentations of isometrically isomorphic L^p spaces is co-3-Σ^0_3-complete.

Theorem (Brown, M. Melnikov 2018+)

The set of all indices of computable presentations of Lebesgue spaces is Π^0_3.

Conclusion

- The complexity of an index sets reflects the complexity of the associated classification problem.
- Index sets are well-studied in computability theory and algebra.
- The study of index sets has recently been extended to analysis.
- We have presented results related to classification of L^p spaces and classes of L^p spaces.
- We know an upper bound on the complexity of the index set of computable presentations of Lebesgue spaces; a lower bound is currently unknown.