A NEW APPROXIMATION FOR EFFECTIVE HAMILTONIANS FOR HOMOGENIZATION OF A CLASS OF HAMILTON-JACOBI EQUATIONS

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Abstract. We propose a new formulation to compute effective Hamiltonians for homogenization of a class of Hamilton-Jacobi equations. Our formulation utilizes an observation made by Barron-Jensen [3] about viscosity supersolutions of Hamilton-Jacobi equations. The key idea is to link the effective Hamiltonian to a suitable effective equation. The main advantage of our formulation is that only one auxiliary equation needs to be solved in order to compute the effective Hamiltonian $\bar{H}(p)$ for all p. Error estimates and stability are proved and numerical examples are presented to demonstrate the performance.

Key words. homogenization, effective Hamiltonian, Hamilton-Jacobi, stability, error estimate, Barron-Jensen theorem

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1. Introduction. Let $\mathbb{T}^n$ be the n dimensional flat torus. Assume that $u^\epsilon \in C(\mathbb{R}^n \times [0, +\infty))$ is the viscosity solution of

$$\begin{cases} u_t^\epsilon + H(Du^\epsilon, \frac{x}{\epsilon}) = 0, \\ u^\epsilon(x, 0) = g(x), \end{cases} \tag{1.1}$$

where $g \in C(\mathbb{R}^n)$ and $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is $\mathbb{T}^n$ periodic in the second variable. Under suitable assumptions, it was proved by Lions, Papanicolaou and Varadhan in [25] that as $\epsilon \to 0$, u^ϵ uniformly converges to the unique viscosity solution u of the effective equation

$$\begin{cases} u_t + \bar{H}(Du) = 0, \\ u(x, 0) = g(x). \end{cases} \tag{1.2}$$

The function $\bar{H} : \mathbb{R}^n \to \mathbb{R}$ is called the effective Hamiltonian, which is defined through the following cell problem [25].

DEFINITION 1.1 (Theorem). *For each $p \in \mathbb{R}^n$, there exists a unique real number $\bar{H}(p)$ such that the partial differential equation (PDE)*

$$H(p + Dv(y), y) = \bar{H}(p) \tag{1.3}$$

has a $\mathbb{T}^n$-periodic viscosity solution v. There has been a lot of interest in developing efficient algorithms to compute $\bar{H}$. Let us mention two motivations.

• The homogenization result in [25] can be used to compute the homogenized solution from the x-independent effective equation (1.2) without the need to resolve the small scale ϵ if $\bar{H}$ is known. We refer to [16] and [1] for instance.

• A program has been launched recently to use nonlinear PDEs to investigate some integrable structures within a dynamical system. And the effective Hamiltonian encodes a lot of dynamical information. We refer to [19, 18, 14] for more backgrounds.

Although the cell problem (1.3) gives an elegant mathematical description of the effective Hamiltonian, it is in general not easy to write out an explicit formula, except for one-dimensional cases [25]. Numerically computing the effective Hamiltonian based on the cell problem is difficult even though there is no small scale involved, which is due to the periodic boundary condition that means there is no way to find out where a characteristic starts and to follow it. Moreover, it requires to solve a cell problem for each single p.

Two well-known algorithms to compute the effective Hamiltonian based on the cell problem are the small-δ method and the large-T method, e.g. in [32] (also Ergodic approximation and long time approximation in [1]). They are essentially the same. The small-δ method uses the fact that under appropriate assumptions, the viscosity solution of an approximate cell problem

$$\delta u^\delta(y) + H(p + Du^\delta(y), y) = 0$$

has the following property:

$$\delta u^\delta \to -\bar{H}(p) \quad \text{as } \delta \to 0 \quad \text{uniformly in } \mathbb{R}^n.$$

The Large-T method uses the result that under appropriate assumptions, the viscosity solution of an evolution problem

$$\begin{cases} u_t + H(p + Du, y) = 0 & \text{in } \mathbb{R}^n \times (0, \ \infty), \\ u = g & \text{in } \mathbb{R}^n \times (t = 0), \end{cases}$$

has the following property:

$$\bar{H}(p) = - \lim_{t \to \infty} \frac{u(y, t)}{t}.$$

When the Hamiltonian is convex in the gradient variable, Gomes and Oberman [23] proposed an approach to calculate $\bar{H}$ based on the inf-max formula (see Contreras-Iturriaga-Paternain-Paternain [9], Gomes [22], etc.)

$$\bar{H}(p) = \inf_{\phi \in C^1(\mathbb{T}^n)} \max_{\mathbb{T}^n} H(p + D\phi, x). \tag{1.4}$$

We also refer to [20] for a numerical scheme based on another variational approach introduced in [17].

In order to compute $\bar{H}(p)$, all the above methods require the solution of the cell problem or a variational problem for each p. Moreover, the method based on the inf-max formula in [23] is computational expensive.

Recently, Oberman, Takei and Vladimirsky [28] proposed an interesting idea to approximate $\bar{H}$ when the Hamiltonian is convex and homogeneous of degree one in the gradient variable. Their basic idea is to recover the effective Hamiltonian from a suitable effective equation. Precisely speaking, for such an H, owing to the inf-max formula, the associated $\bar{H}$ is also convex and homogeneous of degree one. Therefore one may write

$$\bar{H}(p) = \max_{|\alpha|=1}\{(p \cdot \alpha)\bar{c}(\alpha)\},$$

where $\bar{c}(\alpha) \in \mathbb{R}$. Hence to compute $\bar{H}$ it suffices to determine $\bar{c}(\alpha)$. Suppose that u is a viscosity solution of the effective equation

$$
\begin{cases}
\bar{H}(Du) = \max_{|\alpha|=1}\{(Du \cdot \alpha)\bar{c}(\alpha)\} = 1, \\
\\
\qquad\qquad\qquad\qquad u(0) = 0.
\end{cases}
$$

The Hopf-Lax formula implies that

$$
\bar{c}(\alpha) = \frac{1}{u(\alpha)}. \tag{1.5}
$$

And u can be approximated by solutions of the oscillatory equation

$$
\begin{cases}
H(Du^\epsilon, \dfrac{x}{\epsilon}) = \max_{|\alpha|=1}\{(Du^\epsilon \cdot \alpha)c(\dfrac{x}{\epsilon}, \alpha)\} = 1, \\
\\
\qquad\qquad\qquad\qquad u^\epsilon(0) = 0,
\end{cases}
$$

which can be numerically computed by well-established schemes. It is easy to see that by taking proper powers this method actually works for homogeneous Hamiltonian of any degree. The main advantage of this method is that only one auxiliary equation needs to be solved to approximate the effective Hamiltonian for all $p \in \mathbb{R}^n$. However, the assumptions on the Hamiltonian H is too restricted to include many other cases.

In this work, we propose a formulation along this line to include more general Hamiltonians. The main novelty is how to link the effective Hamiltonian to a suitable effective equation. In general, there does not exist an elegant relation like (1.5). Instead, we use an observation made in [3] of viscosity supersolutions for convex Hamilton-Jacobi equations.

Theorem 2.7 in Section 2 is our main theoretical result. It holds for any Hamiltonian $H = H(p, x) \in C(\mathbb{R}^n \times \mathbb{R}^n)$ which is $\mathbb{T}^n$ periodic in the x variable, convex and coercive in the p variable. Coercivity in p means that $H(p, x) \to \infty$ as $|p| \to \infty$ uniformly for $x \in \mathbb{R}^n$. In the Appendix, we also present a formal derivation of an improved error estimate which appears in numerical computations. In order to make our scheme more explicit, throughout this paper, we will focus on Hamiltonian $H(p, x)$ in the kinetic form

$$
H(p, x) = \sum_{1 \le i, j \le n} a_{i,j}(x)p_i p_j + V(x), \tag{1.6}
$$

where $a_{i,j}$ and $V : \mathbb{R}^n \to \mathbb{R}$ are continuous, $\mathbb{T}^n$-periodic functions. And $a_{i,j}(x)$ satisfies the uniformly strict convexity condition

$$
\Lambda|\xi|^2 \ge a_{i,j}(x)\xi_i\xi_j \ge \lambda|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n, \tag{1.7}
$$

with $\Lambda > \lambda > 0$. Hamiltonians in kinetic forms have applications in fields such as the dynamical system and the classical mechanics. Due to the presence of the potential V, the associated effective Hamiltonian is more general and more complicated than those in [28]. For example, it is not homogeneous of any degree and the effective Hamiltonian might contain a flat part near the origin which is related to the trapping of trajectories (see [8] and computational examples in Section 3). Moreover, we also apply our scheme to compute the effective Hamiltonian for a specific convective Hamiltonian $H(p, x) = \frac{1}{2}|p|^2 + b(x) \cdot p$ (see Section 2.1, example 5 and 6 in Section 3).

Outline. In Section 2, we present our approach with theoretical justifications and error estimates. In Section 3, we first present the numerical algorithm to compute effective Hamiltonians. Then numerical results are presented to show both efficiency and accuracy of our method. Complexity of our method and comparisons with other methods are also discussed. In Section 4, we give conclusion remarks and discuss some future projects.

2. Theoretical results. The viscosity solutions, under appropriate assumptions, for Hamilton-Jacobi equations

$$H(Du(x), u, x) = 0, \quad x \in \Omega \subset \mathbb{R}^n$$

are defined as follows [10].

DEFINITION 2.1 (viscosity solution). *A function $u(x) \in C(\Omega)$ is a viscosity subsolution (resp. supersolution) of $H(Du(x), u, x) = 0$ if for any $\phi(x) \in C^\infty(\Omega)$, when $u - \phi$ attains a local maximum (resp. minimum) at point $x_0 \in \Omega$,*

$$H(D\phi(x_0), u(x_0), x_0) \le 0 \quad (resp. \ge 0).$$

A viscosity solution of $H(Du(x), u, x) = 0$ is a viscosity subsolution and supersolution. Our approach is strongly motivated by the following theorem in [3].

THEOREM 2.2 (Barron-Jensen). *If the Hamiltonian $H(p, u, x)$ is convex in p, the inequality in the definition of viscosity supersolutions is actually an equality, that is, for any $\phi(x) \in C^\infty(\Omega)$, when $u - \phi$ attains a local minimum at point $x_0 \in \Omega$,*

$$H(D\phi(x_0), u(x_0), x_0) = 0. \tag{2.1}$$

For readers' convenience, we sketch the proof of Theorem 2.2. For $K \subset \mathbb{R}^n$, denote $Co(K)$ as the convex hull of K, i.e, the smallest closed convex set containing K. We denote $C_r = Co(\{Du(x)| \ x \in B_r(x_0), \ Du(x) \text{ exists}\})$ for $r > 0$. By mollifying u, it is not hard to show that

$$D\phi(x_0) \in \cap_{r>0} C_r.$$

Then the convexity of H in the p variable implies

$$H(D\phi(x_0), u(x_0), x_0) \le 0.$$

Combining with the definition of supersolutions, the equality (2.1) holds.

Barron-Jensen's observation allows us to recover $\bar{H}(p)$ from a suitable effective equation. The following is our approach.

Step I: Consider the effective equation

$$(HJa) \quad \begin{cases} \bar{H}(Du) = f(x) & \text{in } \Omega \backslash \{0\} \subset \mathbb{R}^n, \\ u(0) = 0. \end{cases} \tag{2.2}$$

For each $p \in \mathbb{R}^n$, let $w(x) = u - p \cdot x$. If w attains a local minimum at x_0, then $\bar{H}(p) = f(x_0)$ according to equality (2.1). A tricky point here is that we should choose $f(0) = \min \bar{H}$. Then 0 is a removable source point (see Lemma 2.5 below).

Step II: It might be impossible to find a local minimum point x_0. To overcome this issue, we prove a stability result (Theorem 2.6) which states that if $w(x_1) \leq \min_{\mathbb{R}^n} w + \delta$ for some x_1 and $\delta \geq 0$, then

$$|f(x_1) - \bar{H}(p)| = O(\sqrt{\delta}).$$

Step III: The next task is to find at least one such x_1. Let us turn to the oscillatory equation

$$(HJa)^\epsilon \qquad \begin{cases} H(Du^\epsilon, \dfrac{x}{\epsilon}) = f(x) & \text{in } \Omega \backslash \{0\} \subset \mathbb{R}^n, \\ \\ u^\epsilon(0) = 0. \end{cases} \qquad (2.3)$$

Since $\lim_{\epsilon \to 0} u^\epsilon = u$, we may consider $w_\epsilon = u^\epsilon - p \cdot x$. If $w_\epsilon(x_2) \leq \min_{\mathbb{R}^n} w_\epsilon + \delta$ for some x_2 and $\delta \geq 0$, step II implies

$$|f(x_2) - \bar{H}(p)| \leq O(\sqrt{\max_{B_{R(|p|)}} |u^\epsilon - u| + \delta}),$$

where $R(|p|)$ is a constant that only depends on $|p|$. Formally, $|u^\epsilon - u| = O(\epsilon)$. This suggests that we may expect that

$$|f(x_2) - \bar{H}(p)| \leq O(\sqrt{\epsilon + \delta}).$$

This will be rigorously established in Theorem 2.7 later which is the main theoretical result of our paper. Although Theorem 2.7 is not really based on step I and II, it is strongly motivated by them and Barron-Jensen's Theorem and shares a lot of similarities in proofs. Our formulation works for all H which is convex and coercive in the gradient variable. If $\min \bar{H}$ is known, we have an explicit scheme to compute it. In general, $\min \bar{H}$ is hard to obtain analytically, however we show that for Hamiltonians in kinetic form (1.6) and for convective Hamiltonians (2.5) in Section 2.1, $\min \bar{H}$ is known analytically under appropriate assumptions.

Let us start with (HJa) and $(HJa)^\epsilon$. With appropriate $f(x)$, for any $p \in \mathbb{R}^n$, $u^\epsilon - p \cdot x$ and $u - p \cdot x$ attain at least one global minimum in $\mathbb{R}^n$.

In this work, for Hamiltonians H given in the kinetic forms (1.6), we choose

$$f(x) = 4\Lambda |x|^2 + \max_x V(x) \qquad (2.4)$$

in $(HJa)^\epsilon$ and (HJa), where Λ is the constant as in (1.7).

We present the following theorems and lemmas to verify step I, II and III of our approach. Note that for the kinetic Hamiltonian (1.6), it is easy to deduce from the inf-max formula that

$$\min_{p \in \mathbb{R}^n} \bar{H}(p) = \bar{H}(0) = \max_{\mathbb{T}^n} V.$$

Let us first verify step I.

THEOREM 2.3. *The viscosity solution u^ϵ of $(HJa)^\epsilon$, with Hamiltonian H in the kinetic forms (1.6) and f in (2.4), has the following properties:*
   1. $u^\epsilon(x) \geq |x|^2$ for any $x \in \mathbb{R}^n$;
   2. $|Du^\epsilon|^2 \leq \frac{4\Lambda}{\lambda}|x|^2 + \frac{2}{\lambda}\max_x |V|$.

Therefore, for any $p \in \mathbb{R}^n$, $u^\epsilon - p \cdot x$ attains a local minimum at some point in $\mathbb{R}^n$. And if $u^\epsilon(x) - p \cdot x$ for a fixed $p \in \mathbb{R}^n$ satisfies

$$u^\epsilon(x_0) - p \cdot x_0 = \min_x \{u^\epsilon(x) - p \cdot x\},$$

we have

$$|x_0| \leq |p|.$$

Proof. First we prove $v(x) = |x|^2$ is a viscosity subsolution. Plug $v(x)$ into $(HJa)^\epsilon$, we get

$$a_{i,j}(\frac{x}{\epsilon})2x_i 2x_j + V(\frac{x}{\epsilon}) \leq 4\Lambda|x|^2 + \max_x V = f(x),$$

so $v(x) = |x|^2$ is a smooth viscosity subsolution. According to the comparison principle, $u^\epsilon(x) \geq |x|^2$.

Next we prove part 2.

$$a_{i,j}(\frac{x}{\epsilon})u^\epsilon_{x_i} u^\epsilon_{x_j} + V(\frac{x}{\epsilon}) = 4\Lambda|x|^2 + \max_x V$$

$$\Rightarrow$$

$$a_{i,j}(\frac{x}{\epsilon})u^\epsilon_{x_i} u^\epsilon_{x_j} \leq 4\Lambda|x|^2 + 2\max_x |V|$$

$$\Rightarrow$$

$$\lambda|Du^\epsilon|^2 \leq 4\Lambda|x|^2 + 2\max_x |V|$$

$$\Rightarrow$$

$$|Du^\epsilon|^2 \leq \frac{4\Lambda}{\lambda}|x|^2 + \frac{2}{\lambda}\max_x |V|.$$

Lastly, we see that

$$|x_0|^2 - |p||x_0| \leq x_0 \cdot x_0 - p \cdot x_0 \leq u^\epsilon(x_0) - p \cdot x_0 = \min_x\{u^\epsilon(x) - p \cdot x\} \leq 0,$$

which implies

$$|x_0| \leq |p|.$$

This completes the proof.   □

Through the inf-max formula, for the kinetic Hamiltonian (1.6) it is easy to deduce that

$$\lambda|p|^2 + \min_{\mathbb{T}^n} V \leq \bar{H}(p) \leq \Lambda|p|^2 + \max_{\mathbb{T}^n} V.$$

Following the same proof as in Theorem 2.3, we have the following result.

THEOREM 2.4. *The viscosity solution u of (HJa), with Hamiltonian H in the kinetic forms (1.6) and f in (2.4), has the following properties:*

1. $u(x) \geq |x|^2$ for any $x \in \mathbb{R}^n$;
2. $|Du|^2 \leq \frac{4\Lambda}{\lambda}|x|^2 + \frac{2}{\lambda}\max_x |V|$.

Therefore, for any $p \in \mathbb{R}^n$, $u - p \cdot x$ attains a global minimum at some point in $\mathbb{R}^n$. And if $u(x) - p \cdot x$ for a fixed $p \in \mathbb{R}^n$ satisfies

$$u(x_0) - p \cdot x_0 = \min_x \{u(x) - p \cdot x\},$$

we have

$$|x_0| \leq |p|.$$

REMARK 1. *For general convex and coercive Hamiltonians H, in order to ensure that $u^\epsilon(x) \geq |x|^2$, we can simply choose $f(x)$ such that $f(0) = \min_p \bar{H}(p)$ and*

$$f(x) \geq H(2x, x) \quad \text{for all } x \in \mathbb{R}^n.$$

The coercivity of H controls the growth of $|Du^\epsilon|$. For example, for the convective Hamiltonian $H(p, x) = \frac{1}{2}|p|^2 + b(x) \cdot p$, we may choose $f(x) = 2|x|^2 + C|x| + \min \bar{H}(p)$ or $f(x) = 2|x|^2 + C \sum_{i=1}^n |x_i| + \min \bar{H}(p)$ for $C = \max_{x \in \mathbb{T}^n} |b(x)|$. Also, $\min_p \bar{H}(p) = 0$ if $b(x)$ vanishes at one point or is divergence free and has zero mean. See Section 2.1.

We can further prove that the solution u is actually a viscosity solution in the whole domain including the source point $0 \in \Omega$.

LEMMA 2.5. *Assume that $H(p, x) \in C(\mathbb{R}^n \times \mathbb{R}^n)$ is $\mathbb{T}^n$ periodic in the x variable, convex and coercive in the p variable, and f is Lipschitz continuous and satisfies $f(0) = \min \bar{H}(p)$. Let u be the viscosity solution of (HJa), then u is the viscosity solution in the whole domain Ω. Therefore, for any $p \in \mathbb{R}^n$, if $u - p \cdot x$ attains a local minimum at $0 \in \Omega$, then $\bar{H}(p) = f(0) = \min \bar{H}$.*

Proof. First we show that u is a viscosity supersolution. We only need to consider the source point 0. For any $\phi \in C^1(\Omega)$ and $u - \phi$ attains a local minimum at $0 \in \Omega$, we need to show that $\bar{H}(D\phi(0)) \geq f(0)$. This is automatically true since $f(0) = \min_p \bar{H}(p)$, which implies

$$\bar{H}(D\phi(0)) \geq f(0).$$

Next we prove that u is a viscosity subsolution. This follows from a result in Barron-Jensen [3] which states that if H is convex in the gradient variable, then u is a viscosity subsolution of $H(Du, x) = 0$ in Ω if and only if

$$H(Du, x) \leq 0 \quad \text{for a.e } x \in \Omega.$$

This completes the proof. □

REMARK 2. *Theorem 2.3 and 2.4 verify that $u^\epsilon(x) - p \cdot x$ and $u(x) - p \cdot x$ can obtain at least one local or global minimum at certain point x_0, and $|x_0| \leq |q|$. Lemma 2.5 shows that for $u(x) - p \cdot x$, if $x_0 = 0$, the equality in Barron-Jensen's Theorem also holds since u is proved to satisfy the definition of viscosity solutions at 0. Therefore, step I is verified. Note that Lemma 2.5 is proved for general convex and coercive Hamiltonians.*

Next, we present the following stability result to verify step II.

THEOREM 2.6. *Assume that $H(p, x) \in C(\mathbb{R}^n \times \mathbb{R}^n)$ is $\mathbb{T}^n$ periodic in the x variable, convex and coercive in the p variable, and f is Lipschitz continuous and satisfies $f(0) = \min \bar{H}(p)$. Denote $w(x) = u(x) - p \cdot x$, where u is the viscosity solution of (HJa). If for some point $x_1 \in \Omega$ such that*

$$u(x_1) - p \cdot x_1 = w(x_1) \leq \min_x w(x) + \delta = \min_x \{u(x) - p \cdot x\} + \delta,$$

for $\delta \geq 0$, then,

$$\bar{H}(p) = f(x_1) + O(\sqrt{\delta}).$$

Proof. If u is the viscosity solution of (HJa), we denote

$$g(x) = u(x) - p \cdot x - (\delta - 2|x - x_1|^2).$$

We show that $g(x)$ attains a local minimum in $B(x_1, \sqrt{\delta})$.

We have

$$g(x_1) = u(x_1) - p \cdot x_1 - \delta \leq \min_x w(x),$$

and

$$g(x) = u(x) - p \cdot x + \delta > \min_x w(x) \ \text{ for any } \ x \in \partial B(x_1, \sqrt{\delta}).$$

Therefore $g(x)$ attains a local minimum at some point $x_0 \in B(x_1, \sqrt{\delta})$, which implies $u(x) - (p \cdot x + (\delta - 2|x - x_1|^2))$ attains a local minimum at $x_0 \in B(x_1, \sqrt{\delta})$. According to Barron-Jensen's observation and lemma above,

$$\bar{H}(p - 4(x_0 - x_1)) = f(x_0),$$

which implies

$$\bar{H}(p) + O(|x_0 - x_1|) = f(x_1) + O(|x_0 - x_1|),$$

so we have

$$\bar{H}(p) = f(x_1) + O(|x_0 - x_1|) = f(x_1) + O(\sqrt{\delta}).$$

This completes the proof.   □

Suppose we want to compute $\bar{H}(p)$ for $|p| \leq M$ with M being a fixed positive constant, Theorem 2.3 and 2.4 allow us to restrict the discussion and computation within the cube $\Omega = [-M, \ M]^n$. Let us denote $e_M = \max_\Omega |u^\epsilon - u|$, then according to Theorem 2.6, if

$$u^\epsilon(x_0) - p \cdot x_0 \leq \min_x \{u^\epsilon(x) - p \cdot x\} + \delta,$$

for some x_0 in Ω and $\delta > 0$, then

$$\bar{H}(p) = f(x_0) + O(\sqrt{e_M + \delta}).$$

It is a subtle question that how fast the error $e_M \to 0$ as $\epsilon \to 0$. Heuristically, $\epsilon_M = O(\epsilon)$ due to the formal asymptotic expansion $u^\epsilon = u + \epsilon v(x, \frac{x}{\epsilon})$, where $v = v(x, y)$ satisfies

$$H(Du(x) + D_y v(x, y), y) = \bar{H}(Du(x)).$$

However, it is usually not easy to derive such an estimate rigorously except in some special cases (for example, when u is a constant as in [13]). The main reason is that v might not be regular enough to justify the above expansion. Nevertheless, through

a different approach, we are able to establish the following result which is our main theorem to compute $\bar{H}$. And it verifies step III.

THEOREM 2.7. *Assume that $H(p, x) \in C(\mathbb{R}^n \times \mathbb{R}^n)$ is $\mathbb{T}^n$ periodic in the x variable, convex and coercive in the p variable, and f is Lipschitz continuous and satisfies $f(0) = \min \bar{H}(p)$. Let u^ϵ be the viscosity solution of $(HJa)^\epsilon$, given $p \in \mathbb{R}^n$ such that $|p| \le M$ for some constant $M > 0$ and*

$$u^\epsilon(x_0) - p \cdot x_0 \le \min_x \{u^\epsilon(x) - p \cdot x\} + \delta$$

for some x_0 in Ω and $\delta > 0$, then

$$\bar{H}(p) = f(x_0) + O(\sqrt{\epsilon + \delta}),$$

where the coefficient of $O(\cdot)$ depends on M and the Hamiltonian H.

Proof. For convenience, we assume that $H \in C^\infty$ and is strictly convex in the p variable. Such assumptions can be easily removed by mollification and approximation. Let $w \in W^{1,\infty}(\mathbb{T}^n)$ be a viscosity solution of

$$-H(p + Dw(y), y) = -\bar{H}(p) \quad \text{in } \mathbb{T}^n$$

and satisfy that $w(0) = 0$. The existence of w is similar to Definition 1 (the cell problem). See also [21]. The coercivity of H with respect to p implies that $\|w\|_{W^{1,\infty}(\mathbb{T}^n)} \le C$ for a constant $C \ge 1$ which only depends on M and H. Denote $\tau = \delta + \epsilon$ and

$$v_\epsilon(x) = p \cdot x + \epsilon w(\frac{x}{\epsilon}) - (2C + 1)|x - x_0|^2.$$

Note that

$$u^\epsilon(x_0) - v_\epsilon(x_0) \le \min_{\mathbb{R}^n}(u^\epsilon - p \cdot x) + C\tau$$

and

$$u^\epsilon(x) - v_\epsilon(x) \ge \min_{\mathbb{R}^n}(u^\epsilon - p \cdot x) + (C + 1)\tau \quad \text{if } |x - x_0| = \sqrt{\tau}.$$

Hence there must exist $\bar{x} \in B_{\sqrt{\tau}}(x_0)$ such that

$$u^\epsilon(\bar{x}) - v_\epsilon(\bar{x}) = \min_{\bar{B}_{\sqrt{\tau}}}(u^\epsilon - v_\epsilon).$$

Since a_{ij} and V are all smooth, according to [24] and [21], u^ϵ is semiconcave in $\mathbb{R}^n \backslash \{0\}$ and w is semiconvex in $\mathbb{R}^n$.

Case 1: If $\bar{x} \ne 0$, then both u^ϵ and v_ϵ are differentiable at $\bar{x}$. Hence

$$H(p + Dw(\frac{\bar{x}}{\epsilon}) - 2(2C + 1)(\bar{x} - x_0), \frac{\bar{x}}{\epsilon}) = f(\bar{x}).$$

Since $|\bar{x} - x_0| \le \sqrt{\tau}$ and

$$H(p + Dw(\frac{\bar{x}}{\epsilon}), \frac{\bar{x}}{\epsilon}) = \bar{H}(p),$$

we derive that

$$|f(x_0) - \bar{H}(p)| = O(\sqrt{\tau}).$$

Case 2: If $\bar{x} = 0$, as in the proof of Barron-Jensen's Theorem, we first mollify u^ϵ. Then by comparing the mollification with the semiconvex function v_ϵ, it is not hard to show that there exists a sequence $x_m \to 0$ as $m \to +\infty$ such that v_ϵ is differentiable at x_m and

$$H(p + Dw(\frac{x_m}{\epsilon}) - 2(2C+1)(x_m - x_0), \frac{x_m}{\epsilon}) \leq f(x_m) + o(1),$$

where $o(1) \to 0^+$ as $m \to +\infty$. Again, since

$$H(p + Dw(\frac{x_m}{\epsilon}), \frac{x_m}{\epsilon}) = \bar{H}(p),$$

we have that

$$\bar{H}(p) \leq f(0) + O(\sqrt{\tau}) = \min_{\mathbb{R}^n} \bar{H} + O(\sqrt{\tau}).$$

Accordingly,

$$|\bar{H}(p) - f(0)| \leq O(\sqrt{\tau}).$$

Note that $|x_0| = |x_0 - \bar{x}| \leq \sqrt{\tau}$. So

$$|\bar{H}(p) - f(x_0)| \leq O(\sqrt{\tau}).$$

$\square$

REMARK 3. *The δ in Theorem 2.7 could be the numerical error in computing u^ϵ. So a good guideline for the balance between numerical error for computing u^ϵ, denoted by $\delta(h)$ for a mesh size h, and the asymptotic error is $\delta(h) \sim \epsilon$, which will be used in our numerical examples in Section 3. Note that Theorem 2.6 and 2.7 are proved for general convex and coercive Hamiltonians.*

When $\delta = 0$, the error estimate from the above theorem is $\sqrt{\epsilon}$. However, computational examples suggest that the optimal error estimate might be $O(\epsilon)$. When $n = 1$ we are able to derive this. For kinetic Hamiltonians (1.6), according to [25, 21], the effective Hamiltonian $\bar{H}$ is explicitly given by

$$\begin{cases} \bar{H}(p) = \max_{\mathbb{T}^1} V, & \text{if } |p| \leq p_0, \\ |p| = \int_0^1 \sqrt{2\bar{H}(p) - 2V(y)}\, dy, & \text{otherwise,} \end{cases}$$

where

$$p_0 = \int_0^1 \sqrt{2(\max_{\mathbb{T}^1} V - V(y))}\, dy.$$

Without loss of generality, we may assume that $\max_{\mathbb{T}^1} V = 0$.

THEOREM 2.8. *Suppose that $n = 1$, $\bar{H}(p) > \max_{\mathbb{T}^1} V$ and u^ϵ is the viscosity solution of $(HJa)^\epsilon$ with Hamiltonian H in the kinetic forms (1.6) and f in (2.4). If for some $x_0 \in \mathbb{R}^1$ such that*

$$u^\epsilon(x_0) - p \cdot x_0 = \min_{\mathbb{R}^1}\{u^\epsilon - p \cdot x\},$$

then,

$$|f(x_0) - \bar{H}(p)| \leq O(\epsilon).$$

Proof. Without loss of generality, we assume that $p > 0$, then for $x \geq 0$,

$$u^\epsilon(x) = \int_0^x \sqrt{2f(y) - 2V(\frac{y}{\epsilon})}\, dy.$$

Since $u^\epsilon - px$ attains a minimum at x_0, for any $\Delta x > 0$, we have

$$\int_{x_0}^{x_0 + \Delta x} \sqrt{2f(y) - 2V(\frac{y}{\epsilon})}\, dy \geq p\Delta x.$$

Choosing $\Delta x = \epsilon$ and $x = \frac{y}{\epsilon}$, we derive that

$$\int_{x_\epsilon}^{x_\epsilon + 1} \sqrt{2f(\epsilon x) - 2V(x)}\, dx \geq p,$$

where $x_\epsilon = \frac{x_0}{\epsilon}$. According to Theorem 2.7, $f(x_0) > \max_{\mathbb{T}^1} V$ when ϵ is small. Hence it is clear that

$$\int_{x_\epsilon}^{x_\epsilon + 1} \sqrt{2f(\epsilon x) - 2V(x)}\, dx = \int_{x_\epsilon}^{x_\epsilon + 1} \sqrt{2f(x_0) - 2V(x)}\, dx + O(\epsilon).$$

Since V is periodic,

$$p = \int_0^1 \sqrt{2\bar{H}(p) - 2V(x)}\, dx = \int_{x_\epsilon}^{x_\epsilon + 1} \sqrt{2\bar{H}(p) - 2V(x)}\, dx.$$

Then we have

$$\int_{x_\epsilon}^{x_\epsilon + 1} \sqrt{2f(x_0) - 2V(x)}\, dx \geq \int_{x_\epsilon}^{x_\epsilon + 1} \sqrt{2\bar{H}(p) - 2V(x)}\, dx - O(\epsilon).$$

This implies

$$f(x_0) \geq \bar{H}(p) - O(\epsilon).$$

Similarly, we can deduce that

$$f(x_0) \leq \bar{H}(p) + O(\epsilon)$$

by considering the inequality

$$\int_{x_0 - \Delta x}^{x_0} \sqrt{2f(y) - 2V(\frac{y}{\epsilon})}\, dy \leq p\Delta x,$$

for $\Delta x = \epsilon$. $\square$

REMARK 4. *Theorem 2.8 can be generalized to general strictly convex Hamiltonians in 1-d. For each x, let $H(p_0(x), x) = \min_p H(p, x)$. Since H is strictly convex in p, $p_0(x)$ is unique for each x. Let $F_\pm(q, x)$ be the inverse function of $H(p, x)$ in p for $p > (<)p_0(x)$ respectively, i.e. $F_\pm(H(p, x), x) = p$. If $f(x)$ is chosen as discussed in Remark 1, for $x > 0$ we have*

$$u^\epsilon(x) = \int_0^x F_+(f(y), \frac{y}{\epsilon})\, dy.$$

For a fixed p, $\bar{H}(p) > \max_x \inf_p H(p,x)$, we take the viscosity solution of the cell problem for the same branch as for u^ϵ and have

$$p = \int_0^1 F_+(\bar{H}(p), y)\, dy.$$

As long as H is Lipschitz continuous in p, we can use the same argument as above to show that $|f(x_0) - \bar{H}(p)| = O(\epsilon)$.

We conjecture that the above theorem should also hold for $n > 1$ in proper sense. However, when $n > 1$, the situation becomes much more complicated due to the lack of explicit expression for $\bar{H}(p)$. In Appendix A, we show a formal derivation on $O(\epsilon)$ error estimate for $n > 1$.

2.1. $\min \bar{H}(p)$ for convective Hamiltonians. In this section, we study another class of Hamiltonians where $\min \bar{H}(p)$ can be found analytically. We look at the convective Hamiltonian

$$H(p,x) = \frac{1}{2}|p|^2 + b(x) \cdot p, \tag{2.5}$$

where $b(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathbb{T}^n$-periodic smooth vector function. According to the model proposed in [15], the corresponding effective Hamiltonian represents the *turbulent flame speed* from the turbulent combustion theory. For general velocity field b, $\min \bar{H}(p)$ is unknown except the following two cases.

THEOREM 2.9. *Assume that the Hamiltonian H is given by (2.5). Suppose that one of the following holds,*

(a) there exists a point x_0 such that $b(x_0) = 0$,

(b) the flow is incompressible and mean zero, i.e. $\nabla \cdot b = 0$ and $\int_{\mathbb{T}^n} b(x)\, dx = 0$,

then

$$\min \bar{H}(p) = \bar{H}(0) = 0. \tag{2.6}$$

Proof. We first show that (a) $\Rightarrow$(2.6). Since $b(x_0) = 0$, for $p \in \mathbb{R}^n$ and $\phi \in C^1$,

$$\max_{\mathbb{T}^n}\{\frac{1}{2}|p + D\phi|^2 + b(x) \cdot D\phi\} \geq \frac{1}{2}|p + D\phi(x_0)|^2 \geq 0.$$

Accordingly, by the inf-max formula

$$\bar{H}(p) = \inf_{\phi \in C^1(\mathbb{T}^2)} \max_{\mathbb{T}^n}\{\frac{1}{2}|p + D\phi|^2 + b(x) \cdot (p + D\phi)\} \geq 0.$$

And by the inf-max formula, we also have

$$\bar{H}(0) = \inf_{\phi \in C^1(\mathbb{T}^2)} \max_{\mathbb{T}^n}\{\frac{1}{2}|D\phi|^2 + b(x) \cdot D\phi\} \leq 0. \tag{2.7}$$

The last equality comes from choosing $\phi \equiv 0$. Hence (2.6) holds.

Next we show that (b)$\Rightarrow$ (2.6). We prove that $\bar{H}(p) \geq \frac{|p|^2}{2}$, which physically means that an incompressible flow will enhance flame propagation. In fact, for $p \in \mathbb{R}^n$, let u be a solution of the cell problem

$$\frac{1}{2}|p + Du|^2 + b(x) \cdot (p + Du) = \bar{H}(p). \tag{2.8}$$

Since $\nabla \cdot b = 0$ and $\int_{\mathbb{T}^n} b \, dx = 0$, integration by parts leads to

$$\int_{\mathbb{T}^n} b(x) \cdot (p + Du) = 0.$$

Hence taking integration on both sides of (2.8), we get that

$$\bar{H}(p) \geq \frac{1}{2} \int_{\mathbb{T}^n} |p + Du|^2 \, dx = \frac{1}{2} \int_{\mathbb{T}^n} |Du|^2 \, dx + \frac{|p|^2}{2} \geq \frac{|p|^2}{2}.$$

Combining with (2.7), we get (2.6).   □

There are two examples of incompressible flows that appear often in the literature. One is the shear flow (e.g. $b(x) = (v(x_2), 0)$ for $x = (x_1, x_2)$ and appropriate v). An explicit formula of $\bar{H}(p)$ is given in [15]. Another one is the cellular flow (see [27] for instance) and there is no explicit formula of $\bar{H}(p)$. In Section 3 , we compute $\bar{H}(p)$ for specific cellular flow and shear flow.

REMARK 5. *Note that the velocity flow $b(x)$ is time-independent, which is a special case for the velocity flow considered in [15], where the velocity flow is time-periodic. Our method focuses on the time-independent cases.*

2.2. A scaling property of $\bar{H}$ for the kinetic Hamiltonian. Suppose we want to compute $\bar{H}(p)$ for $|p| \leq M$ with $M > 0$ some fixed constant, and Hamiltonian H and f being in (1.6) and (2.4) respectively, Theorem 2.3 and 2.4 show that we need to solve $(HJa)^\epsilon$ on a cubic domain $\Omega = [-M, \ M]^n$. If M is large, solving $(HJa)^\epsilon$ numerically could be expensive. Here we point out a simple scaling property of $\bar{H}$ to resolve this issue. This scaling property allows us to avoid solving $(HJa)^\epsilon$ on a large domain.

The cell problems

$$H(Du + p, x) = a_{i,j}(x)(u_{x_i} + p_i)(u_{x_j} + p_j) + V(x) = \bar{H}(p)$$

and

$$H(\frac{Du + p}{M}, x) = a_{i,j}(x)\frac{(u_{x_i} + p_i)}{M}\frac{(u_{x_j} + p_j)}{M} + \frac{V(x)}{M^2} = \frac{\bar{H}(p)}{M^2}$$

are equivalent.

Let us define $F(p, x) = a_{i,j}(x)p_i p_j + \frac{V(x)}{M^2}$, then we have

$$F(Dv, x) = a_{i,j}(x)v_{x_i} v_{x_j} + \frac{V(x)}{M^2}, \quad \text{with } v(x) = \frac{u(x)}{M}.$$

Through the cell problem (Definition 1.1), we have

$$F(Dv + \frac{p}{M}, x) = \bar{F}(\frac{p}{M}).$$

And we see that

$$F(Dv + \frac{p}{M}, x) = H(\frac{Du + p}{M}, x) = \frac{\bar{H}(p)}{M^2},$$

which implies

$$\bar{H}(p) = M^2 \bar{F}(\frac{p}{M}).$$

Therefore, in order to get $\bar{H}(p)$, we only need to compute $\bar{F}(\frac{p}{M})$ with small $\frac{p}{M}$, which requires to solve $(HJa)^\epsilon$ with Hamiltonian F on a much smaller domain.

REMARK 6. *Theoretically, if we can solve $(HJa)^\epsilon$ on the whole space $\mathbb{R}^n$, then we can obtain $\bar{H}(p)$ for all $p \in \mathbb{R}^n$. However, solving $(HJa)^\epsilon$ on the whole space $\mathbb{R}^n$ is numerically impossible. Fortunately, for many problems like (1.2), if $|Dg| \leq M$ for some $M > 0$, then it is easy to show that under appropriate assumptions, $|Du(x,t)| \leq M$ for all $(x,t) \in \mathbb{R}^n \times [0,+\infty)$. Therefore, we can focus on $\bar{H}(p)$ for $|p| \leq M$, and Theorem 2.3 and 2.4 show that we only need to solve $(HJa)^\epsilon$ on domain $\Omega = [-M, \ M]^n$, which can be done numerically.*

And when M is large, we may utilize the scaling property to reduce the computational expenses if it is applicable. Note that the scaling property also holds for convective Hamiltonians in Section 2.1.

3. Numerical method and examples. Compared to other numerical methods for computing effective Hamiltonians, which are surveyed briefly in Section 1, there are two main computational advantages for this new formulation: (1) only one auxiliary equation needs to be solved to obtain $\bar{H}(p)$ for all p (e.g. Remark 6), and (2) the periodic boundary condition for the cell problem is replaced by a regular boundary condition at the removable source point.

In this section we first present our numerical method to compute the effective Hamiltonians. Then we test a few computational examples and use careful error analysis to verify our method. For notational simplicity, the method is illustrated with examples in one or two dimensions, but generalization to higher dimension is straightforward. Complexity and accuracy for computing the effective Hamiltonians are discussed.

The first step in our numerical method is to solve the oscillatory Hamilton-Jacobi equation $(HJa)^\epsilon$ to obtain a numerical approximation u_h^ϵ, where h is the mesh size. Then for each $p \in \mathbb{R}^N$, we find the minimum of $u_h^\epsilon - p \cdot x$ approximately at a mesh point x_0. Finally, $\bar{H}(p)$ is approximated by Theorem 2.7.

3.1. Numerical solution of Hamilton-Jacobi equations. Although the equation $(HJa)^\epsilon$ contains fast oscillations, it is a convex Hamilton-Jacobi equation with a given source point, for which several fast algorithms with optimal complexity are available. In this work we use the fast sweeping method [4, 38] to solve $(HJa)^\epsilon$. The fast sweeping method is an efficient iterative method using Gauss-Seidel iteration with monotone upwind scheme and alternating orderings. The method is very easy to implement, especially for general convex Hamilton-Jacobi equation [30] and has the optimal complexity $O(N)$, where N is the number of grid points. Although the constant in the complexity estimate for the original fast sweeping method depends on how fast characteristics change directions [31], which implies that the number of iterations for $(HJa)^\epsilon$ depends on ϵ, the improvement for the fast sweeping method proposed in [2] using locking and queuing techniques can be adopted, which can dramatically reduce the computational time for $(HJa)^\epsilon$ where fast oscillations appear. As shown in [2], the CPU time is almost always linear in N with a fixed constant no matter how fast the oscillations are. We point out that other fast algorithms such as fast marching method [36, 34] for isotropic Eikonal equation and ordered upwind method [35] for anisotropic Eikonal equation can also be used here. Once $(HJa)^\epsilon$ is solved, one can approximate the effective Hamiltonians $\bar{H}(p)$ for all p easily using our formulation.

3.2. Numerical error and computation complexity. The total approximation error to the effective Hamiltonian is composed of asymptotic error $|u - u^\epsilon|$ that depends on ϵ and the numerical error $|u^\epsilon - u^\epsilon_h|$ that depends on the grid size h. Theoretically, the total error is $O(\sqrt{\epsilon + \delta(h)})$ according to Theorem 2.7, where $\delta(h) = |u^\epsilon - u^\epsilon_h|$. A good balance is to have numerical error $\delta(h)$ comparable to ϵ. However, the above theoretical estimate usually provides a conservative lower bound in practice. Numerical results suggest that the total error is $O(\epsilon)$, which is proved for one dimensional case in Theorem 2.8. In high dimension, a formal proof is given in the Appendix.

Numerically, although we need to solve u^ϵ from $(HJa)^\epsilon$, which contains fast oscillations, the key point is that we only need to approximate $u(x)$, which is the solution to the effective equation (HJa) without fast oscillations, to the order of ϵ and approximate the cell solution $v(\frac{x}{\epsilon})$ to $O(1)$ according to the asymptotic expansion

$$u^\epsilon(x, \frac{x}{\epsilon}) = u(x) + \epsilon v(x, \frac{x}{\epsilon}) + \cdots. \tag{3.1}$$

Here we also give a formal complexity analysis for our method and the large-T method explained in Section 1. Suppose explicit first order Eulerian scheme is used in the large-T method to solve the time-dependent Hamilton-Jacobi equation (e.g. [1]), and we want to compute $\bar{H}(p)$ to the accuracy $O(\gamma)$ in dimension n.

- For our method, numerically the asymptotic error is $\sim \epsilon \sim \gamma \Rightarrow \epsilon \sim \gamma$. In order to have the numerical error of the same order, one needs to solve the fast oscillatory solution v in (3.1) with $O(1)$ error. It requires (formally) $\frac{h}{\epsilon^2} \sim O(1) \Rightarrow h \sim \gamma^2$. Due to the linear complexity for fast sweeping method, the total complexity for our method is $\sim \gamma^{-2n}$.

- For the large-T method, the asymptotic error is $\sim \frac{1}{T} \sim \gamma \Rightarrow T \sim \gamma^{-1}$. The numerical error (formally) is $\sim hT$. However, since $\bar{H} \approx -\frac{u(x,T)}{T}$, we have $h \sim \gamma$. The number of time steps due to standard CFL condition for hyperbolic problem is $\sim \frac{T}{\triangle t} \sim \frac{T}{h} \sim \gamma^{-2}$. Assume that $\bar{H}$ is C^2, in order to find $\bar{H}$ for all p with accuracy $\sim \gamma$, we can first find the $\bar{H}$ on a mesh for p with resolution $\sim \sqrt{\gamma}$, then use the linear interpolation to obtain $\bar{H}$ for all p. The total complexity is $\sim \gamma^{-n}\gamma^{-2}\gamma^{\frac{-n}{2}} \sim \gamma^{-\frac{3n}{2}-2}$. Note that $\gamma^{-2n} < \gamma^{-\frac{3n}{2}-2}$ for $n \leq 3$.

- Heuristically, $\bar{H}^\epsilon(p) = \bar{H}(p) + a_1(p)\epsilon + a_2(p)\epsilon^2 + \cdots$, where $\bar{H}^\epsilon$ denotes the numerical solution obtained by our method. We may employ the Richardson extrapolation to improve the accuracy. For example, $2\bar{H}^{\frac{\epsilon}{2}}(p) - \bar{H}^\epsilon(p)$ is applied to improve accuracy as shown in Remark 7. For our method, the computational complexity is basically the same to apply the Richardson extrapolation. Richardson extrapolation can also be applied in large-T method to improve accuracy, for example by using $2\bar{H}^T(p) - \bar{H}^{\frac{T}{2}}(p)$. Note that, one must refine the resolution for p before using the linear interpolation in order to keep the improved accuracy for all p, which will increase the complexity dramatically.

We can see that the gain is due to the linear complexity of our fast algorithm to solve Hamilton-Jacobi equation with a simple boundary condition. Or equivalently, the large-T method can be viewed as a simple Jacobi type of iterative method to solve the cell problem with a periodic boundary condition, which is not efficient due to the CFL condition.

3.3. Computational Examples. We present a few 1-d and 2-d examples to demonstrate our new method. In particular we show computational study of convergence as $\epsilon \to 0$. As discussed above, we choose our grid size $h = \dfrac{\epsilon}{W}$ for some number W independent of ϵ. For $|p| \leq M$ with $M > 0$ being fixed, Theorem 2.3 and 2.4 allow us to restrict the computation within the cube $\Omega = [-M, \ M]^n$. If $|p|$ is large we show some examples using the scaling argument presented in Section 2.2. Except for 2-d example 5 and 6 with H being the convective Hamiltonian (2.5), the Hamiltonians for other examples are all in kinetic forms (1.6).

3.3.1. 1-d Example. The Hamiltonian is

$$H(p, x) = \frac{1}{2}p^2 + \cos(2\pi x), \quad f(x) = 2x^2 + 1.$$

$\bar{H}(p)$ is exactly known [25]. We will compute $\bar{H}(2) = 2.0637954$.

First we show the accuracy for the fast sweeping method for solving $(HJa)^\epsilon$. The computational domain is $[0, \ 1]$. The grid size h is chosen to resolve ϵ with $h = \dfrac{\epsilon}{20}$. Table 3.1 shows the maximum error of the fast sweeping method. The numerical solution is converging by $O(\epsilon)$.

Error of fast sweeping method for $(HJa)^\epsilon$ in domain $[0, \ 1]$						
Mesh (ϵ, $h = \frac{\epsilon}{20}$)	$\epsilon = \frac{1}{2}$	$\epsilon = \frac{1}{4}$	$\epsilon = \frac{1}{8}$	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$
Error $|u^\epsilon - u_h^\epsilon|$	0.0307068	0.0162982	0.0084036	0.0042489	0.0021060	0.0010217

TABLE 3.1

1-d example: the exact solution $\int_0^1 \sqrt{2(f(x) - \cos(\frac{2\pi x}{\epsilon}))}dx$ is approximated with a much smaller $h = 10^{-5}$.

Next we compute $\bar{H}(2)$ with our new method. The computational domain for solving $(HJa)^\epsilon$ is $[-2, \ 2]$. We choose $h = \dfrac{\epsilon}{20}$. Table 3.2 shows the maximum error, which appears to be $O(\epsilon)$.

Error of $\bar{H}(2)$			
Mesh (ϵ, $h = \frac{\epsilon}{20}$)	$\epsilon = \frac{1}{10}$	$\epsilon = \frac{1}{1000}$	$\epsilon = \frac{1}{100000}$
Error of $\bar{H}(2)$	0.0269954	0.00033012	0.000003

TABLE 3.2

1-d example: accuracy of numerically computing $\bar{H}(2)$.

3.3.2. 2-d Examples. We present a few 2-d examples to illustrate our method.
Example 1: The Hamiltonian is

$$H(p, q, x, y) = \frac{1}{2}(p^2 + q^2) - \frac{1}{2}((2 + \cos(2\pi x)\sin(2\pi y))^2 + (\sin(2\pi x)\cos(2\pi y))^2),$$

$$f(x, y) = 2(x^2 + y^2) - \frac{1}{2}.$$

Especially, $\bar{H}(2, 0) = 0$. We verify our method by computing $\bar{H}(2, 0)$. The computational domain is set on $[-2, \ 2] \times [-2, \ 2]$. We choose $h = \dfrac{\epsilon}{10}$. Table 3.3 shows the maximum error, which appears to be $O(\epsilon)$.

Numerical error of $\bar{H}(2,0)$, exact $\bar{H}(2,0) = 0$					
Mesh (ϵ, $h = \frac{\epsilon}{10}$)	$\epsilon = \frac{1}{2}$	$\epsilon = \frac{1}{4}$	$\epsilon = \frac{1}{8}$	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$
Error	-0.235000	-0.133750	-0.070937	-0.036484	-0.018496

TABLE 3.3

2-d example 1: Numerical error for $\bar{H}(2,0)$.

Example 2: The Hamiltonian is

$$H(p,q,x,y) = \frac{1}{2}(p^2 + q^2) - \cos(2\pi x) - \cos(2\pi y), \ f(x,y) = 2(x^2 + y^2) + 2.$$

Especially, $\bar{H}(1.4006, 1.1205) = 2.1241$. Table 3.4 shows the maximum error for computing $\bar{H}(1.4006, 1.1205)$ with our new method. We choose computational domain to be $[-2, \ 2] \times [-2, \ 2]$ and $h = \frac{\epsilon}{10}$. The numerical solution appears to converge by $O(\epsilon)$. We would like to mention that this example was also calculated in [1] using the Large-T method, with a semi-implicit scheme. By choosing $T = 500$, $\Delta t = 1$ and $h = \frac{1}{200}$, the error was 0.0014 which is almost the same as what we get when $\epsilon = \frac{1}{128}$. For computing $\bar{H}(1.4006, 1.1205)$, our method takes 8 iterations to converge for computing u_h^ϵ with the standard fast sweeping method. So the overall complexity is of $8 \times O(5120^2) + O(5120^2)$. The second part is the complexity for finding the global minimum. The Large-T methods in [1] takes $500 \times O(200^2)$ operations to get $\bar{H}(1.4006, 1.1205)$. Although CFL condition for time step is relaxed by using semi-implicit scheme, a large system of linear equations has to be solved at each time step. So the coefficient of $O(200^2)$ depends on the method used to solve the large system of linear equations. More importantly, our method can compute various p easily once the $(HJa)^\epsilon$ is solved.

Numerical error of $\bar{H}(1.4006, 1.1205)$, exact $\bar{H}(1.4006, 1.1205) = 2.1241$					
Mesh (ϵ, $h = \frac{\epsilon}{10}$)	$\epsilon = \frac{1}{8}$	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$	$\epsilon = \frac{1}{128}$
Error	0.022537	0.014334	0.007596	0.003568	0.001389

TABLE 3.4

2-d example 2: Numerical error for $\bar{H}(1.4006, 1.1205)$.

Example 3: The Hamiltonian is

$$H(p,q,x,y) = \frac{1}{2}(p^2 + q^2) + \cos(2\pi x) + \cos(2\pi y) + \cos(2\pi(x-y)), \ f(x,y) = 2(x^2 + y^2) + 3.$$

Figure 3.1 shows the plots of $\bar{H}(p)$. We choose $\epsilon = \frac{1}{20}$ and $h = \frac{\epsilon}{10}$. The computational domain is $[-4, \ 4] \times [-4, \ 4]$. The domain of $\bar{H}(p)$ is $[-4, \ 4] \times [-4, \ 4]$ with a resolution of 81×81 points.

The flat region in Figure 3.1 (blank region in contour plot) corresponds to $\bar{H}(p) = \max_{x,y} V(x,y) = \max_{x,y} \cos(2\pi x) + \cos(2\pi y) + \cos(2\pi(x-y)) = 3$. Simple computation with Large-T method in [32] by forward Euler first order Godunov scheme gives $\bar{H}(-4,4) = 16.0400$ with spatial discretization $h = 0.1$, time step $\Delta t = 0.001$ and terminal time $T_f = 100$. It is close to the one computed with our method, which gives $\bar{H}(-4,4) = 16.0682$.

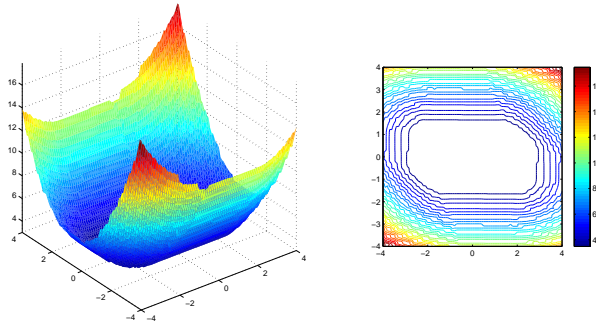**Example 4: Double pendulum**. The Hamiltonian for the double pendulum is

$$H(p,q,x,y) = \frac{1}{2}\frac{p^2 - 2pq\cos(2\pi(x-y)) + 2q^2}{2 - \cos^2(2\pi(x-y))} + 2\cos(2\pi x) + \cos(2\pi y),$$
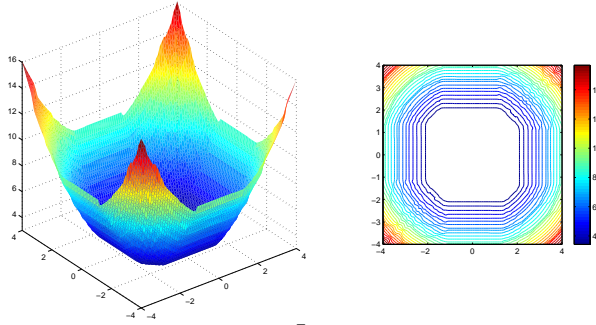
FIG. 3.1. *2-d example 3: $\bar{H}(p)$–surf plot and contour plot.*

with

$$f(x) = 6(x^2 + y^2) + 3.0.$$

Figure 3.2 shows the plots of $\bar{H}(p)$. We choose $\epsilon = \dfrac{1}{20}$ and $h = \dfrac{\epsilon}{10}$. The computational domain is $[-4,\ 4] \times [-4,\ 4]$. The domain of $\bar{H}(p)$ is $[-4,\ 4] \times [-4,\ 4]$ with a resolution of 81×81 points.

FIG. 3.2. *2-d example 4: $\bar{H}(p)$–surf plot and contour plot*

The flat region in Figure 3.2 (blank region in contour plot) corresponds to $\bar{H}(p) = \max_{x,y} V(x,y) = \max_{x,y} 2\cos(2\pi x) + \cos(2\pi y) = 3$. Simple computation with Large-T method in [32] by forward Euler first order Godunov scheme gives $\bar{H}(4,4) = 18.0970$ with spatial discretization $h = 0.1$, time step $\triangle t = 0.001$ and terminal time $T_f = 100$. It is close to the one computed with our method, which gives $\bar{H}(4,4) = 17.9532$.

Example 5: Convective Hamiltonian–shear flow. The Hamiltonian is given by (2.5) with $b(x,y) = (\sin 2\pi y, 0)$. In this case, $\bar{H}(1,0) = 1.5$ (e.g. [15]). According to Remark 1, we choose $f(x) = 2(x^2 + y^2) + 2(|x| + |y|)$.

The numerical discretization of the Hamilton-Jacobi equation can be derived from the general framework developed in [30], which was proved to be equivalent to the control interpretation in [26].

Figure 3.3 shows the plots of $\bar{H}(p)$. We choose $\epsilon = \dfrac{1}{20}$ and $h = \dfrac{\epsilon}{10}$. The computational domain is $[-4,\ 4] \times [-4,\ 4]$. The domain of $\bar{H}(p)$ is $[-4,\ 4] \times [-4,\ 4]$

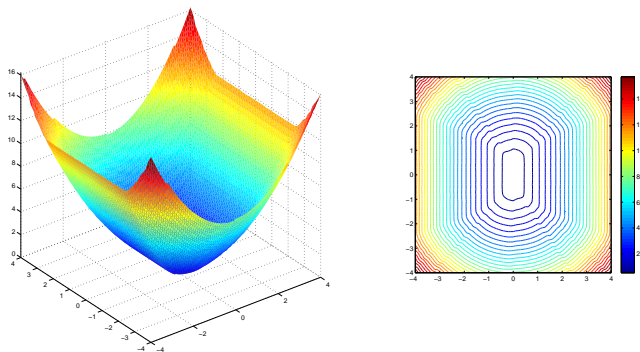with a resolution of 81×81 points. Especially, we get $\bar{H}(1,0) = 1.44065$ with error 0.05935.

FIG. 3.3. *2-d example 5 (shear flow).* $\bar{H}(p)$*–surf plot and contour plot*

Example 6: Convective Hamiltonian–cellular flow. The Hamiltonian is given by (2.5) with $b(x,y) = (-h_y, h_x)$, where $h(x,y) = \sin 2\pi x \sin 2\pi y$. According to Remark 1, we choose $f(x) = 2(x^2 + y^2) + 4\pi(|x| + |y|)$.

Figure 3.4 shows the plots of $\bar{H}(p)$. We choose $\epsilon = \dfrac{1}{20}$ and $h = \dfrac{\epsilon}{10}$. The computational domain is $[-4, \ 4] \times [-4, \ 4]$. The domain of $\bar{H}(p)$ is $[-4, \ 4] \times [-4, \ 4]$ with a resolution of 81×81 points.
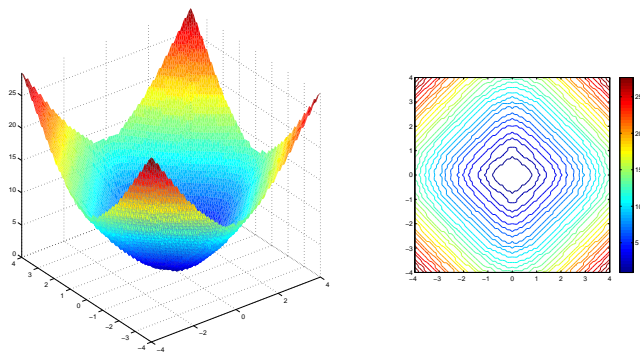

FIG. 3.4. *2-d example 6 (cellular flow).* $\bar{H}(p)$*–surf plot and contour plot*

REMARK 7. *Note that Richardson extrapolation can be applied to improve the accuracy. For instance, in 2-d example 1 and 2-d example 2, we obtain more accurate results by applying Richardson extrapolation, which are shown in Table 3.5 and 3.6.*
Note that 2-d example 1 clearly exhibits the $O(\epsilon^2)$ pattern.

3.3.3. Examples of $\bar{H}(p)$ with large p. In this part, we compute $\bar{H}(100)$ for the above 1-d example and $\bar{H}(100, 100)$ for the above 2-d Example 2. We rescale the problem as discussed in Section 2.2 by choosing $M = 100$. Therefore we only need to compute $\bar{F}(1)$ and $\bar{F}(1,1)$ respectively.

Numerical error of $\bar{H}(2,0)$, exact $\bar{H}(2,0) = 0$					
Mesh (ϵ, $h = \frac{\epsilon}{10}$)	$\epsilon = \frac{1}{2}$	$\epsilon = \frac{1}{4}$	$\epsilon = \frac{1}{8}$	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$
Error	-0.235000	-0.133750	-0.070937	-0.036484	-0.018496
Richardson extrapolation	-0.032500	-0.008124	-0.002031	-0.000508	

TABLE 3.5

2-d example 1: Numerical error for $\bar{H}(2,0)$ with Richardson extrapolation.

Numerical error of $\bar{H}(1.4006, 1.1205)$, exact $\bar{H}(1.4006, 1.1205) = 2.1241$					
Mesh (ϵ, $h = \frac{\epsilon}{10}$)	$\epsilon = \frac{1}{8}$	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$	$\epsilon = \frac{1}{128}$
Error	0.022537	0.014334	0.007596	0.003568	0.001389
Richardson extrapolation	0.006131	0.000858	0.000461	0.000790	

TABLE 3.6

2-d example 2: Numerical error for $\bar{H}(1.4006, 1.1205)$ with Richardson extrapolation

1-d Example: The Hamiltonian and rescaled Hamiltonian are $H(p, x) = \frac{1}{2}|p|^2 + \cos(2\pi x)$ and $F(p, x) = \frac{1}{2}|p|^2 + \frac{\cos(2\pi x)}{M^2}$ respectively. We choose $f(x) = 2x^2 + \frac{1}{M^2}$, compute $\bar{F}(1)$, and get $\bar{H}(100) = M^2 \bar{F}(1)$. Table 3.7 shows the results.

$\bar{H}(100)$			
Mesh (ϵ, $h = \frac{\epsilon}{20}$)	$\epsilon = \frac{1}{4}$	$\epsilon = \frac{1}{8}$	$\epsilon = \frac{1}{16}$
$\bar{H}(100)$	5001.00	4876.78	4938.70

TABLE 3.7

1-d example: computing $\bar{H}(100)$ with rescaling.

2-d Example: The Hamiltonian and rescaled Hamiltonian are $H(p, q, x, y) = \frac{1}{2}(p^2 + q^2) - \cos(2\pi x) - \cos(2\pi y)$ and $F(p, q, x, y) = \frac{1}{2}(p^2 + q^2) - \dfrac{\cos(2\pi x) + \cos(2\pi y)}{M^2}$ respectively.

We choose $f(x) = 2x^2 + \frac{2}{M^2}$, compute $\bar{F}(1, 1)$, and get $\bar{H}(100, 100) = M^2 \bar{F}(1, 1)$. Table 3.8 shows the results.

For the above two examples, theoretically, $\bar{H}(100, 100)$ (2-d) is the sum of $\bar{H}(100) + \bar{H}(100)$ (1-d) dimension by dimension, which is verified as shown in Table 3.7 and 3.8.

4. Conclusion and future projects. We present a new approximation for the effective Hamiltonians, which has accuracy and efficiency verified both theoretically and numerically. The key point is that only one auxiliary PDE needs to be solved for all p. Unfortunately, our method can not deal with nonconvex Hamiltonians, since the "equality" for the definition of viscosity supersolution is no longer valid. However it already covers a wide class of applications with convex Hamiltonians involved. Besides, different formulations with the same idea can be used, which we remark here.

For example, instead of using the Eikonal type equation $(HJa)^\epsilon$ and (HJa), we can also employ the following PDEs $(HJb)^\epsilon$ and (HJb) as in infinite horizon problems,

$$(HJb) \qquad \begin{cases} \bar{H}(Du) + u = f(x) & \text{in } \Omega \backslash \{0\} \in \mathbb{R}^n, \\ \quad u(0) = 0, \end{cases} \qquad (4.1)$$

$\bar{H}(100, 100)$			
Mesh (ϵ, $h = \frac{\epsilon}{20}$)	$\epsilon = \frac{1}{4}$	$\epsilon = \frac{1}{8}$	$\epsilon = \frac{1}{16}$
$\bar{H}(100, 100)$	9508.25	9753.56	9877.39

TABLE 3.8

2-d example: computing $\bar{H}(100, 100)$ with rescaling.

and the corresponding oscillatory equation

$$(HJb)^{\epsilon} \qquad \begin{cases} H(Du^{\epsilon}, \dfrac{x}{\epsilon}) + u^{\epsilon}(x) = f(x) & \text{in } \Omega \backslash \{0\} \subset \mathbb{R}^n, \\ u^{\epsilon}(0) = 0. \end{cases} \qquad (4.2)$$

All procedures are similar except that we need to use $f(x_0) - u(x_0)$ instead of $f(x_0)$ to calculate $\bar{H}(p)$ if $u - p \cdot x$ attains local minimum at x_0. One advantage of using (HJb) and $(HJb)^{\epsilon}$ is the possibility to derive some rigorous error estimates between u^{ϵ} and u_h^{ϵ} as in [7, 11, 12]. Meanwhile, we would like to point out that theoretical error estimates are usually too conservative compared to the real numerical errors, therefore an ongoing project is to prove $O(\epsilon)$ error estimate in proper sense for $n > 1$ as in Theorem 2.8, the situation is much more complicated due to the lack of explicit relation between $\bar{H}(p)$ and the potential V for $n > 1$. In Appendix A, we show a formal discussion.

As we mentioned in the introduction, one of the motivations to compute $\bar{H}$ is to provide a more efficient way to solve equations for general initial data g. Another ongoing project is to find a good approach to solve equations based on $\bar{H}$ derived with our method. Similar work has been done in [1] using the large-T method.

We also would like to extend our method to deal with convex but noncoercive Hamiltonians, for example, the G-equation $G_t + |DG| + b(x) \cdot DG = 0$. The convective Hamiltonian $H(p, x) = |p| + b(x) \cdot p$ is convex but not coercive when the velocity field b is too large. The corresponding effective Hamiltonian has been used in the combustion theory to model the *turbulent flame speed* [33, 29] (also see [37, 5] for periodic homogenization of G-equation, and [6] for a Lagrangian method on G-equation).

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Appendix A. Formal discussion of $O(\epsilon)$ error estimate for $n > 1$.

In this section we present a formal proof of $O(\epsilon)$ error estimate for $n > 1$. Let us assume the following,

1. $u \in C^2(\mathbb{R}^n)$ and there exists a unique point $\bar{x} \in \mathbb{R}^n$ such that

$$u(\bar{x}) - p \cdot \bar{x} = \min_{\mathbb{R}^n}\{u(x) - p \cdot x\},$$

that is, the plane $p \cdot x$ is a supporting plane for the viscosity solution u at $\bar{x}$ (e.g. Figure A.1). Note that this formal proof can be easily extended to the case when $u(x) - p \cdot x$ attains minimum at finitely many points.
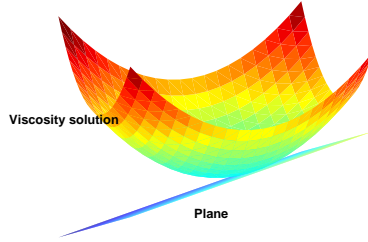

FIG. A.1. *Supporting plane.*

2. $D^2 u(\bar{x})$ is non-degenerate, i.e.,

$$D^2 u(\bar{x}) > 0.$$

3. $u^\epsilon(x) = u(x) + \epsilon v(x, \frac{x}{\epsilon})$, where $v(x, y)$ is C^1 and periodic in the y variable.

Denote $w(x) = u(x) - p \cdot x$ and $w^\epsilon(x) = u^\epsilon(x) - p \cdot x$. Then there exist λ, Λ and $\delta > 0$ such that when ϵ is small enough, we have

$$w^\epsilon(\bar{x}) < w^\epsilon(x) \quad \text{if} \quad |x - \bar{x}| \geq \delta, \tag{A.1}$$

$$\lambda|x - \bar{x}|^2 \leq w(x) - w(\bar{x}) \leq \Lambda|x - \bar{x}|^2 \quad \text{if} \quad |x - \bar{x}| \leq \delta. \tag{A.2}$$

We prove that there exists some constant K independent of ϵ such that if

$$w^\epsilon(x_\epsilon) = \min_{\mathbb{R}^n} w^\epsilon(x)$$

for some $x_\epsilon \in \mathbb{R}^n$, then

$$|x_\epsilon - \bar{x}| \leq K\epsilon.$$

Proof. By (A.1), it is easy to see that $|x_\epsilon - \bar{x}| < \delta$. Denote

$$L = \max_{\bar{B}_\delta(\bar{x})} |v_x(x, y)|,$$

and choose $K > \sqrt{n}$ such that

$$\lambda K^2 - KL > \Lambda n + L\sqrt{n},$$

we show that

$$|x_\epsilon - \bar{x}| \le K\epsilon.$$

Let us write $x = \bar{x} + \epsilon y$ and assume that

$$\min_{y \in \mathbb{T}^n} v(\bar{x}, y) = \alpha,$$

then for $y \in Q_1 = \{y = (y_1, y_2, ..., y_n) : |y_i| \le 1 \text{ for } i = 1 : n\}$,

$$w^\epsilon(\bar{x} + \epsilon y) = w(\bar{x} + \epsilon y) + \epsilon v(\bar{x} + \epsilon y, \frac{\bar{x}}{\epsilon} + y)$$
$$\le w(\bar{x}) + \epsilon v(\bar{x}, \frac{\bar{x}}{\epsilon} + y) + (\Lambda n + L\sqrt{n})\epsilon^2.$$

Accordingly, we have

$$\min_{y \in Q_1} w^\epsilon(\bar{x} + \epsilon y) \le w(\bar{x}) + (\Lambda n + L\sqrt{n})\epsilon^2 + \epsilon \alpha.$$

If $|x - \bar{x}| > K\epsilon$, i.e. $|y| > K$, then we have

$$w^\epsilon(\bar{x} + \epsilon y) = w(\bar{x} + \epsilon y) + \epsilon v(\bar{x} + \epsilon y, \frac{\bar{x}}{\epsilon} + y)$$
$$\ge w(\bar{x}) + \epsilon v(\bar{x}, \frac{\bar{x}}{\epsilon} + y) + (\lambda|y|^2 - L|y|)\epsilon^2$$
$$> w(\bar{x}) + \epsilon v(\bar{x}, \frac{\bar{x}}{\epsilon} + y) + (\Lambda n + L\sqrt{n})\epsilon^2$$
$$\ge \min_{y \in Q_1} w^\epsilon(\bar{x} + \epsilon y).$$

Therefore, we have

$$|x_\epsilon - \bar{x}| \le K\epsilon.$$

□