A Criterion for Parameter Identification in Structural Equation Models

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Abstract

This paper deals with the problem of identifying direct causal effects in recursive linear structural equation models. The paper establishes a sufficient criterion for identifying individual causal effects and provides a procedure computing identified causal effects in terms of observed covariance matrix.

1 Introduction

Structural equation models (SEMs) have dominated causal reasoning in the social sciences and economics, in which interactions among variables are usually assumed to be linear [Duncan, 1975, Bollen, 1989]. This paper deals with one fundamental problem in SEMs, accessing the strength of linear cause-effect relationships from a combination of observational data and model structures.

The problem has been under study for half a century, primarily by econometricians and social scientists, under the name "The Identification Problem"[Fisher, 1966]. Although many algebraic or graphical methods have been developed, the problem is still far from being solved. In other words, we do not have a necessary and sufficient criterion for deciding whether a causal effect can be computed from observed data. Most available methods are sufficient criteria which are applicable only when certain restricted conditions are met.

In this paper, we provide a new sufficient criterion for the identification of individual causal effects. Our method is based on the partial regression equations, which reduce the identification problem into a problem of solving a set of algebraic equations [Tian, 2005]. We present a procedure that will determine sufficient conditions under which these equations can be solved and express identified causal effects in terms of observed covariances.

We begin with an introduction to SEMs and the identification problem, and give a brief review to previous work before presenting our results. Due to space constraints, the proofs are not given. Proofs of all results are given in [Tian, 2007].

2 Linear SEMs and Identification

A linear SEM over a set of random variables $V = \{V_1, \ldots, V_n\}$ is given by a set of structural equations of the form

$$V_j = \sum_i c_{ji} V_i + \epsilon_j, \quad j = 1, \ldots, n, \tag{1}$$

where the summation is over the variables in V judged to be immediate causes of V_j. c_{ji}, called a *path coefficient*, quantifies the direct causal influence of V_i on V_j, and is also called a *direct effect*. ϵ_j's represent "error" terms and are assumed to have normal distribution. In this paper we consider recursive models and assume that the summation in Eq. (1) is for $i < j$, that is, $c_{ji} = 0$ for $i \geq j$. The set of variables (and the corresponding structural equations) are considered to be ordered as $V_1 < V_2 < \ldots < V_n$. We denote the covariances between observed variables $\sigma_{ij} = Cov(V_i, V_j)$, and between error terms $\psi_{ij} = Cov(\epsilon_i, \epsilon_j)$. We denote the following matrices, $\Sigma = [\sigma_{ij}]$, $\Psi = [\psi_{ij}]$, and $C = [c_{ij}]$. Without loss of generality, the model is assumed to be standardized such that each variable V_j has zero mean and variance 1.

The structural assumptions encoded in the model are the zero path coefficient c_{ji}'s and zero error covariance ψ_{ij}'s. The model structure can be represented by a directed acyclic graph (DAG) G with (dashed) bidirected edges, called a *causal diagram* (or *path diagram*), as follows: the nodes of G are the variables $V_1, \ldots, V_n$; there is a directed edge from V_i to V_j in G if V_i appears in the structural equation for V_j, that is, $c_{ji} \neq 0$; there is a bidirected edge between V_i and V_j if the error terms ϵ_i and ϵ_j have non-zero correlation ($\psi_{ij} \neq 0$). Figure 1 shows a simple SEM and the corresponding causal diagram in which each directed edge is annotated by the corresponding path coefficient.

The parameters of the model are the non-zero entries in the

$X = \epsilon_x$
$W = aX + \epsilon_w$
$Z = bX + \epsilon_z$
$Y = cW + dZ + \epsilon_y$
$Cov(\epsilon_x, \epsilon_z) \neq 0$
$Cov(\epsilon_w, \epsilon_y) \neq 0$

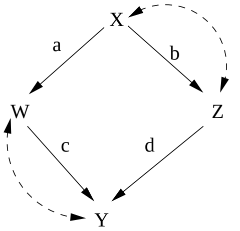


Figure 1: A linear SEM.

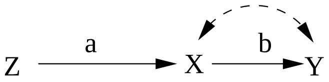


Figure 2: A typical instrumental variable

matrices C and Ψ. Fixing the model structure and given parameters C and Ψ, the covariance matrix Σ is given by (see, for example, [Bollen, 1989])

$$\Sigma = (I - C)^{-1}\Psi(I - C)^{t^{-1}}. \quad (2)$$

Conversely, in the identification problem, given the structure of a model, one attempts to solve for C in terms of the given observed covariance matrix Σ. If Eq. (2) gives a unique solution to some path coefficient c_{ji}, independent of the (unobserved) error correlations Ψ, the path coefficient c_{ji} is said to be *identified*; otherwise, c_{ji} is said to be *non-identifiable*. In other words, the *identification problem* is that whether a path coefficient is determined uniquely from the covariance matrix Σ given the causal diagram. If every parameter of the model is identified, then *the model is identified*. Note that the identifiability conditions we seek involve the structure of the model alone, not particular numerical values of parameters, that is, we insist on having *identification almost everywhere*, allowing for pathological exceptions (see, for example, [Brito and Pearl, 2002a] for formal definition of identification almost everywhere).

3 Previous Work

Many methods have been developed for deciding whether a specific parameter or a model is identifiable. For example, the well-known instrumental variable (IV) method [Bowden and Turkington, 1984] requires searching for variables (called *instruments*) that are uncorrelated with the error terms in specific equations. A typical configuration of the IV method is shown in Fig. 2, in which Z serves as an instrument for identifying the causal effect b as

$$b = \sigma_{ZY}/\sigma_{ZX}. \quad (3)$$

Traditional approaches are based on algebraic manipulation of the structural equations [Fisher, 1966, Bekker *et al.*, 1994, Rigdon, 1995]. In recent years graphical approaches for identifying linear causal effects have been developed, and some sufficient graphical conditions for identification were established [McDonald, 1997, Pearl, 1998, Spirtes *et al.*, 1998, Pearl, 2000, Spirtes *et al.*, 2001, Tian, 2004]. The applications of these methods are limited in scope, and typically some special conditions have to be met for these methods to be applicable.

One principled approach for the identification problem is to write Eq.(2) for each term σ_{ij} of Σ using Wright's method of path coefficients [Wright, 1934]. Wright's equations consist of equating the (standardized) covariance σ_{ij} with the sum of products of parameters (c_{kl}'s and ψ_{kl}'s) along certain paths between V_i and V_j in the causal diagram. For example, the Wright's equations for the model in Fig. 2 are

$$\sigma_{ZX} = a \quad (4)$$

$$\sigma_{ZY} = ab \quad (5)$$

$$\sigma_{XY} = b + \psi_{XY}. \quad (6)$$

Then a path coefficient c_{ij} is identified if and only if Wright's equations give a unique solution to c_{ij}, independent of error correlation ψ_{kl}'s.

Based on Wright's equations, a number of sufficient graphical criteria for identification have been developed. [Brito and Pearl, 2002b, Brito and Pearl, 2006] defined "auxiliary set" to obtain a set of linearly independent equations with path coefficients c_{jk}'s as unknowns. However, the coefficients of these equations are functions of other parameters which must be identified before solving the equations for c_{jk}'s. So they established sufficient criteria for *model identification*, that is, conditions for *every* parameter in the model to be identified. [Brito and Pearl, 2002a] defined "instrumental set" also obtaining a set of linearly independent equations with path coefficients as unknowns. The approach provides sufficient conditions for identifying individual path coefficients. The main difficulty seems to be that it is not an easy task to find out an instrumental set.

Recently, another principled approach for the identification problem is presented in [Tian, 2005], in which the partial regression coefficients, instead of the covariance, are expressed in terms of model parameters obtaining so called partial regression equations. Then a path coefficient c_{ij} is identified if and only if the set of partial regression equations give a unique solution to c_{ij}, independent of ψ_{kl}'s.

In this paper, we will derive sufficient conditions for identifying individual path coefficients based on partial regression equations. We acknowledge that some of the ideas in this paper were inspired by the work in [Brito and Pearl, 2002c, Brito and Pearl, 2002b, Brito and Pearl, 2002a, Brito and Pearl, 2006]. First we introduce the partial regression equations.

4 Partial Regression Equations

For a set $S \subseteq V$, let $\beta_{ij.S}$ denote the *partial regression coefficient* which represents the coefficient of V_j in the linear regression of V_i on V_j and S. (Note that the order of the subscripts in $\beta_{ij.S}$ is essential.) Partial regression coefficients can be expressed in terms of covariance matrices as follows [Cramer, 1946]:

$$\beta_{ij.S} = \frac{\Sigma_{V_iV_j} - \Sigma^T_{V_iS}\Sigma^{-1}_{SS}\Sigma_{V_jS}}{\Sigma_{V_jV_j} - \Sigma^T_{V_jS}\Sigma^{-1}_{SS}\Sigma_{V_jS}}, \tag{7}$$

where Σ_{SS} etc. represent covariance matrices over corresponding variables.

Let S_{jk} denote a set

$$S_{jk} = \{V_1, \ldots, V_{j-1}\} \setminus \{V_k\}. \tag{8}$$

[Tian, 2005] derived an expression for the partial regression coefficient $\beta_{jk.S_{jk}}$ in terms of the model parameters (path coefficients and error covariances) given by

$$\beta_{jk.S_{jk}} = c_{jk} + \alpha_{jk} - \sum_{k+1 \le l \le j-1} \beta_{lk.S_{lk}} \alpha_{jl},$$
$$j = 2, \ldots, n, \; k = 1, \ldots, j-1, \tag{9}$$

where α_{jk}'s are defined during the process of "orthogonalizing" the set of error terms to obtain a new set of error terms $\{\epsilon'_1, \ldots, \epsilon'_n\}$ that are mutually orthogonal in the sense that

$$Cov(\epsilon'_i, \epsilon'_j) = 0, \quad \text{for } i \neq j. \tag{10}$$

The Gram-Schmidt orthogonalization process proceeds recursively as follows. We set

$$\epsilon'_1 = \epsilon_1. \tag{11}$$

For $j = 2, \ldots, n$, we set

$$\epsilon'_j = \epsilon_j - \sum_{k=1}^{j-1} \alpha_{jk} \epsilon'_k, \tag{12}$$

in which

$$\alpha_{jk} = \frac{Cov(\epsilon_j, \epsilon'_k)}{Cov(\epsilon'_k, \epsilon'_k)}. \tag{13}$$

The set of equations given by (9) are called the *partial regression equations*. As an example, the partial regression equations for the model shown in Figure 1 are given by

$$\beta_{WX} = a \tag{14}$$
$$\beta_{ZW.X} = 0 \tag{15}$$
$$\beta_{ZX.W} = b + \alpha_{ZX} \tag{16}$$
$$\beta_{YZ.WX} = d \tag{17}$$
$$\beta_{YW.XZ} = c + \alpha_{YW} \tag{18}$$
$$\beta_{YX.WZ} = -\beta_{WX}\alpha_{YW} \tag{19}$$

It is not difficult to solve these equations to obtain that the path coefficients a, d, and c are identified.

In general, given the model structure (represented by zero path coefficients and zero error correlations), some of the c_{jk}'s and α_{jk}'s will be set to zero in Eq. (9), and we can solve the identifiability problem by solving Eq. (9) for c_{jk}'s in terms of the partial regression coefficients. This provides a principled method for solving the identifiability problem. A path coefficient c_{ij} is identified if and only if the set of partial regression equations give a unique solution to c_{ij}, independent of error correlations.

The partial regression equations are linear with respect to path coefficient c_{jk}'s and parameter α_{jk}'s, but may not be linear with respect to ψ_{ij}'s. α_{jk}'s are nonlinear functions of ψ_{ij}'s and may not be independent with each other. However, we can treat α_{jk}'s as independent unknowns and solve the partial regression equations (9). If we find a unique solution to c_{jk}, then c_{jk} is identified. Although we can not say c_{jk} is nonidentifiable if we could not find a unique solution to c_{jk}. In this paper, we will treat α_{jk}'s as independent unknowns and look for sufficient conditions for identifying path coefficients. The identification problem is reduced to that of solving the set of linear equations (9) for c_{jk}'s in terms of the partial regression coefficients $\beta_{jk.S_{jk}}$'s, and a path coefficient c_{jk} is identified if the set of equations give a unique solution to c_{jk} that is independent of α_{jk}'s.

5 Identifying Path Coefficients

Assume that we want to identify the path coefficients associated with a variable V_j, c_{jk}'s. All we need to do is to solve the $j-1$ equations given in (9) for $k = 1, \ldots, j-1$. We will name each of the equations in (9) after the corresponding variable as in the following:

$$(V_k) : \beta_{jk.S_{jk}} = c_{jk} + \alpha_{jk} - \sum_{k+1 \le l \le j-1} \beta_{lk.S_{lk}} \alpha_{jl}. \tag{20}$$

Let $V_j^< = \{V_1, \ldots, V_{j-1}\}$ denote the set of variables ordered ahead of V_j. Let Pa_j be the set of variables V_k such that $c_{jk} \neq 0$, that is, there is a directed edge from $V_k \to V_j$ in the causal diagram. Let $NP_j = V_j^< \backslash Pa_j$. α_{jk}'s are nonlinear functions of ψ_{ij}'s and some of α_{jk}'s will be identically zero depending on the model structure. Let Ne_j be the set of variables V_k such that α_{jk} is not identically zero denoted by $\alpha_{jk} \not\equiv 0$. In summary, we classify the set of variables in $V_j^<$ into possibly overlapping groups:

$$Pa_j = \{V_k | V_k \in V_j^<, c_{jk} \neq 0\}. \tag{21}$$
$$NP_j = \{V_k | V_k \in V_j^<, c_{jk} = 0\}. \tag{22}$$
$$Ne_j = \{V_k | V_k \in V_j^<, \alpha_{jk} \not\equiv 0\}. \tag{23}$$

Let $E(S)$ denote the set of equations (V_k) such that $V_k \in S$. Each equation (V_k) in $E(Pa_j)$ can be solved for the path

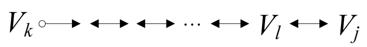


Figure 3: An active path between V_k and V_j given S_{jk}

coefficient c_{jk} by simply rewriting the equation to obtain

$$c_{jk} = \beta_{jk.S_{jk}} - \alpha_{jk} + \sum_{k+1\leq l\leq j-1} \beta_{lk.S_{lk}}\alpha_{jl}, \quad V_k \in Pa_j. \quad (24)$$

We see that c_{jk} is identifiable if each α_{ji}, $i = k, \ldots, j-1$, is either zero or identifiable. And we have the following interesting result.

Proposition 1 $c_{jk} = \beta_{jk.S_{jk}}$ *if* $\alpha_{ji} = 0$, $i = k, \ldots, j-1$.

The graphical condition for α_{ji} being zero is given in Lemma 2 in Section 5.1. Therefore we have a graphical condition under which c_{jk} can be estimated by $\beta_{jk.S_{jk}}$.

To identify α_{ji}'s we need to solve the set of equations in $E(NP_j)$ with α_{ji}'s as unknowns, rewritten in the following

$$(V_k) : \beta_{jk.S_{jk}} = \alpha_{jk} - \sum_{k+1\leq l\leq j-1} \beta_{lk.S_{lk}}\alpha_{jl}, \quad V_k \in NP_j. \quad (25)$$

In general we may have more equations than unknowns, or more unknowns than equations. These equations may not be linearly independent with each other. The difficulty of solving these linear equations lies in that the coefficients of these equations, the partial regression coefficients, are not independent free parameters. The partial regression coefficients are related to each other in a complicated way, and it is difficult to decide the rank of the set of linear equations since it is not easy to determine whether certain expressions of partial regression coefficients will cancel out each other and become identically zero.

To overcome this difficulty, it can be shown that a partial regression coefficient $\beta_{jk.S_{jk}}$ can be expressed in terms of the free parameters c_{il}'s and ψ_{il}'s along certain paths between V_j and V_k in the causal diagram. Then we are able to establish graphical conditions for the linear independence of the set of equations in $E(NP_j)$. First, we define some graphical notations.

5.1 Graph Background

A *path* between two nodes X and Y in a causal diagram consists of a sequence of consecutive edges of any type (directed or bidirected) (we will assume every variable appears only once in the path). A non-endpoint node Z on a path is called a *collider* if two arrowheads on the path meet at Z, i.e. $\rightarrow Z \leftarrow$, $\leftrightarrow Z \leftrightarrow$, $\leftrightarrow Z \leftarrow$, $\rightarrow Z \leftrightarrow$; all other non-endpoint nodes on a path are *non-colliders*, i.e. $\leftarrow Z \rightarrow$, $\leftarrow Z \leftarrow$, $\rightarrow Z \rightarrow$, $\leftrightarrow Z \rightarrow$, $\leftarrow Z \leftrightarrow$. A *bidirected path* is a path such that every edge on the path is bidirected. The following concept of active path plays an important role in the analysis of the partial regression equations.

Definition 1 (Active Path) *A path between two nodes X and Y is said to be* active *given a set of nodes Z if*

(i) every non-collider on the path is not in Z, and

(ii) every collider on the path is in Z or is an ancestor of a node in Z.

The following lemma characterizes a class of active paths.

Lemma 1 *For $k < j$, every node V_l on an active path between V_k and V_j given S_{jk} must be a collider, and $V_l < V_j$ (i.e., $V_l \in S_{jk}$). (see Figure 3)*

The following lemma gives a graphical condition for α_{jk}'s being non-zero.

Lemma 2 *For $k < j$, $\alpha_{jk} \not\equiv 0$ if there is a bidirected edge between V_k and V_j or there is a bidirected path between V_k and V_j such that every intermediate variable V_l on the path is ordered ahead of V_k, $V_l < V_k$; otherwise $\alpha_{jk} = 0$.*

The following proposition, shown in [Pearl, 1998, Spirtes *et al.*, 1998], gives a graphical condition for $\beta_{jk.S_{jk}}$'s being non-zero.

Proposition 2 $\beta_{jk.S_{jk}} = 0$ *if there is no active paths between V_j and V_k given S_{jk}.*

Proposition 2 essentially says that $\beta_{jk.S_{jk}} = 0$ if V_j is conditionally independent of V_k given S_{jk}. With Lemma 2 and Proposition 2, we are now able to write down the partial regression equations by inspecting the causal diagram.

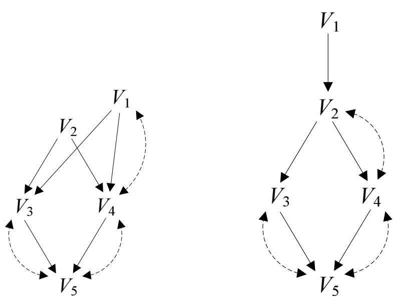
5.2 Accessory Set

To identify α_{ji}'s, we look for a set of variable V_k's such that the set of equation (V_k)'s in (25) give a unique solutions to α_{ji}'s. We will (informally) refer to these variables as accessory variables as they enable the identification of α_{ji}'s. The idea is motivated by the widely used concept of instrumental variables and its extensions (auxiliary variables and instrumental set [Brito and Pearl, 2002b, Brito and Pearl, 2002a, Brito and Pearl, 2006]) which enable the identification of path coefficient c_{ji}'s. Next, we analyse a few examples to see conditions for a variable to serve as an accessory variable.

In the model in Figure 2, Z serves as an instrumental variable for identifying the path coefficient b. In our framework, Z also serves as an accessory variable for identifying α_{YX}, which in turn leads to the identification of b, as shown by the following equations:

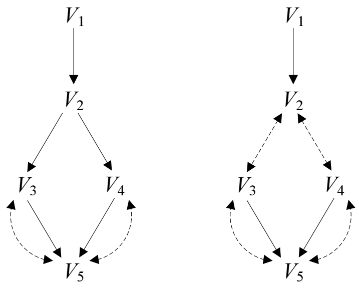
$$\beta_{YX} = b + \alpha_{YX} \quad (26)$$

$$\beta_{YZ.X} = -\beta_{XZ}\alpha_{YX}. \quad (27)$$



(a) (b)

Figure 4: Examples of accessory variables.



(a) (b)

Figure 5: Examples of accessory variables.

In the model in Figure 1, X is not qualified to be an instrumental variable for identifying path coefficient c. However X can serve as an accessory variable for identifying α_{YW} (see Eq. (19)), which then leads to the identification of c (see Eq. (18)).

The key observation seems to be that if there is an active path between V_k and V_j given S_{jk} that ends with $V_i \leftrightarrow V_j$ for $k < i < j$, then α_{ji} will appear in the equation (V_k). In general the equation (V_k) may contain several α_{ji}'s, then we need a set of equation (V_k)'s to simultaneously solve for α_{ji}'s. For example, in the model in Figure 4(a), the equations (V_1) and (V_2) may be solved to identify α_{53} and α_{54} simultaneously:

$$(V_1) : \beta_{51.234} = -\beta_{31.2}\alpha_{53} - \beta_{41.23}\alpha_{54} \quad (28)$$

$$(V_2) : \beta_{52.134} = -\beta_{32.1}\alpha_{53} - \beta_{42.13}\alpha_{54}. \quad (29)$$

In the model in Figure 4(b), the equations (V_1) and (V_2) may also be solved to identify α_{53} and α_{54}. In the model in Figure 5(a), V_2 may serve as an accessory variable for identifying α_{53} and α_{54} but V_1 can not since $\beta_{51.234} = 0$. Therefore we have one equation (V_2) with two unknowns. In the model in Figure 5(b), both V_1 and V_2 can potentially serve as accessory variables, however we can show that the equations (V_1) and (V_2) are not independent.

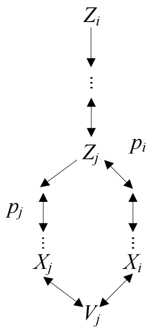


Figure 6: Definition of accessory set.

The conditions for a set of variables V_k's to serve as accessory variables such that the set of equation (V_k)'s are linearly independent are characterized by the following formal definition of accessory variables.

Definition 2 (Accessory Set) *A set of variables* $Z = \{Z_1, \ldots, Z_k\} \subseteq NP_j$ *is said to be an* accessory set *relative to* $X = \{X_1, \ldots, X_k\} \subseteq Ne_j$ *and* V_j *if there exist a set of paths* $p_1, \ldots, p_k$ *such that*

1. *Either* $Z_i = X_i$, p_i *degenerates into a node* X_i*; or* $Z_i < X_i$, *and* p_i *is a path between* Z_i *and* X_i *such that every intermediate node* V_l *is a collider on* p_i *and* $V_l < X_i$ *(i.e.,* p_i *is an active path between* Z_i *and* X_i *given* $S_{X_i Z_i}$*); and*
2. p_i *and* p_j *do not share a common edge, and their only possible common variable must be either*
   - Z_j, *such that* p_i *is a collider at* Z_j *(unless* $Z_j = X_i$, *for this case,* p_i *points at* X_i*),* $Z_j \neq X_j$, *and* p_j *does not point to* Z_j *(see Figure 6); or*
   - Z_i, *such that* p_j *is a collider at* Z_i *(unless* $Z_i = X_j$, *for this case,* p_j *points at* X_j*),* $Z_i \neq X_i$, *and* p_i *does not point to* Z_i.

For example, in Figure 4(a), $\{V_1, V_2\}$ is an accessory set relative to $\{V_3, V_4\}$ and V_5 with paths $V_1 \to V_3$ and $V_2 \to V_4$. In Figure 4(b), $\{V_1, V_2\}$ is an accessory set relative to $\{V_4, V_3\}$ and V_5 with paths $V_1 \to V_2 \leftrightarrow V_4$ and $V_2 \to V_3$ sharing a common variable V_2. In Figure 5(a), $\{V_1, V_2\}$ can not serve as an accessory set relative to $\{V_3, V_4\}$ while $\{V_2\}$ is an accessory set relative to $\{V_3\}$ (or $\{V_4\}$) and V_5. In Figure 5(b), $\{V_1, V_2\}$ can not serve as an accessory set relative to $\{V_3, V_4\}$ while $\{V_1\}$ or $\{V_2\}$ individually is an accessory set relative to $\{V_3\}$ (or $\{V_4\}$) and V_5.

Theorem 1 *If a set of variables* $Z = \{Z_1, \ldots, Z_k\} \subseteq NP_j$ *is an accessory set relative to* $X = \{X_1, \ldots, X_k\} \subseteq Ne_j$ *and* V_j, *then the set of linear equations* $E(Z) = \{(Z_1), \ldots, (Z_k)\}$ *are linearly independent with respect to unknowns* $\alpha_{V_j X_1}, \ldots, \alpha_{V_j X_k}$, *that is,* $E(Z)$ *can be solved with respect to unknowns* $\alpha_{V_j X_1}, \ldots, \alpha_{V_j X_k}$.

An algorithm for finding an accessory set with maximum size is given in Section 6.

5.3 Solving Equations

Assume that we have identified an accessory set $Z = \{Z_1, \ldots, Z_k\}$. Let $U = \{\alpha_{V_j X_1}, \ldots, \alpha_{V_j X_m}\}$ be the set of unknowns appearing in the set of linear equations $E(Z)$. If $k = m$, then by Theorem 1, all the unknowns in U are identified. In general, we may have $m \geq k$, that is, we have more unknowns than equations. One approach for solving the set of equations is to use Simon's causal ordering algorithm [Simon, 1953], extended in [Lu *et al.*, 2000]. Next we describe the algorithm.

We will say that a set of equations S is *self-contained* if the number of equations is equal to the number of unknowns in S. A self-contained set is *minimal* if it does not contain any proper self-contained subsets. By Theorem 1, any self-contained subset S of $E(Z)$ can be solved and all the unknowns in S are identified.

To solve the set of linear equations in $E(Z)$, we start with *identifying* the mininal self-contained subsets in $E(Z)$. Then we *solve* these subsets, remove the equations in those subsets from $E(Z)$, and *substitute* the values of solved variables into remaining equations. The remaining set of equations is called the *derived system*. We keep applying identifying, solving, and substitution operations on derived system until either the derived system D is empty, which happens if $E(Z)$ is self-contained, or there are no more self-contained subsets that can be identified, which means that the set of remaining variables in D can not be solved.

After solving the set of equations $E(Z)$, the identifiability of the path coefficients c_{jk}'s can be determined by Eq. (24).

5.4 An Example

We illustrate the overall process of identifying path coefficients by an example. Consider the model in Figure 7. Assume that we want to identify the path coefficients associated with V_7 (c_{74}, c_{75}, and c_{76}). First we express the path coefficients in terms of α_{7i}'s as follows

$$c_{76} = \beta_{76.12345} - \alpha_{76}. \tag{30}$$

$$c_{75} = \beta_{75.12346} - \alpha_{75}. \tag{31}$$

$$c_{74} = \beta_{74.12356} - \alpha_{74}. \tag{32}$$

Then we search for an accessory set using the algorithm given in Section 6. Assume that the algorithm returns $\{V_1, V_2\}$ as an accessory set. Then we attempt to solve the equations (V_1) and (V_2) given in the following:

$$(V_1) : \beta_{71.23456} = -\beta_{41.23}\alpha_{74} \tag{33}$$

$$(V_2) : \beta_{72.1345} = -\beta_{62.1345}\alpha_{76} - \beta_{52.134}\alpha_{75} - \beta_{42.13}\alpha_{74} \tag{34}$$

(V_1) is self-contained and can be solved to identify α_{74} given by

$$\alpha_{74} = -\beta_{71.23456}/\beta_{41.23}. \tag{35}$$

We then substitute the value of α_{74} into the equation (V_2), and we end up with one equation having two unknowns. Therefore we are not able to identify α_{75} and α_{76}. In general, we can not make claims about being nonidentifiable since what we have presented is merely a sufficient criterion. In this particular model α_{75} and α_{76} are indeed nonidentifiable. Finally, by Eqs. (30)-(32), we conclude that c_{74} is identified and is given by

$$c_{74} = \beta_{74.12356} + \beta_{71.23456}/\beta_{41.23}, \tag{36}$$

and c_{75} and c_{76} are both nonidentifiable. Note that equation (V_3) as given below

$$(V_3) : \beta_{73.12456} = -\beta_{43.12}\alpha_{74}, \tag{37}$$

is not helpful for identifying α_{75} and α_{76}, as expected. In fact (V_3) leads to a constraint on the observed covariance matrix imposed by the model structure if we substitute the solved value of α_{74} into (V_3) obtaining

$$\beta_{73.12456}\beta_{41.23} = \beta_{43.12}\beta_{71.23456}. \tag{38}$$

6 An Algorithm for Finding Accessory Set

In large models, it may not be easy to find an accessory set based on its definition. In this section, we provide an algorithm for finding an accessory set that contains the largest number of variables. We show that the problem can be reduced to a maximum flow problem.

A flow network $F = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$ (see, for example, [Cormen *et al.*, 1990]). We distinguish two vertices in a flow network: a source s and a sink t. A flow in F is a real-valued function $f : V \times V \rightarrow R$ that satisfies the capacity constraints $f(u, v) \leq c(u, v)$ and the flow conservation property (the amount of flow entering any vertex must be the same as the amount of flow leaving the vertex) among others. The value of a flow f is defined as $|f| = \sum_{v \in V} f(s, v)$, that is, the total net flow out of the source. In the maximum flow problem, we are given a flow network F, with source s and sink t, and we wish to find a flow of maximum value from s to t.

To search for an accessory set relative to V_j, we construct a flow network F as follows. The nodes of F consists of:

- for every node $V_i < V_j$, add two nodes V_i^- and V_i^+ into F.
- a source node s.
- a sink node t.

The edges of F are:

- for every node $V_i < V_j$, add edge $V_i^- \to V_i^+$.
- for every edge $V_i \to V_l$, add edge $V_i^- \to V_l^+$.
- for every edge $V_i \leftrightarrow V_l$, add two edges $V_i^+ \to V_l^+$ and $V_i^+ \leftarrow V_l^+$.
- for every node $V_i \in Ne_j$ (those with $\alpha_{V_j X_i} \not\equiv 0$), add edge $V_i^+ \to t$.
- for every node $V_i \in NP_j$ (those with $c_{ji} = 0$), add $s \to V_i^-$.

We assign a capacity 1 to every edge in F. We also assign a node capacity of 1 to every node (except s and t) in F (this can be achieved by splitting every node into two and connecting them by an edge of capacity 1).

We then solve the maximum flow problem on the flow network F (using, for example, Ford-Fulkerson algorithm). Since every edge has a capacity 1 and every node has a capacity 1, the computed flow f represents a set of disjoint directed paths from s to t. Let the set of paths be $q_i = s \to Z_i^- \to \ldots \to X_i^+ \to t$, $i = 1, \ldots, k$, where $k = |f|$. Each path q_i in F is interpreted as corresponding to a path p_i from Z_i to X_i in the causal diagram G as follows:

- An edge $V_a^- \to V_a^+$ corresponds to a node V_a.
- An edge $V_a^- \to V_b^+$ corresponds to an edge $V_a \to V_b$.
- An edge $V_a^+ \to V_b^+$ corresponds to an edge $V_a \leftrightarrow V_b$.

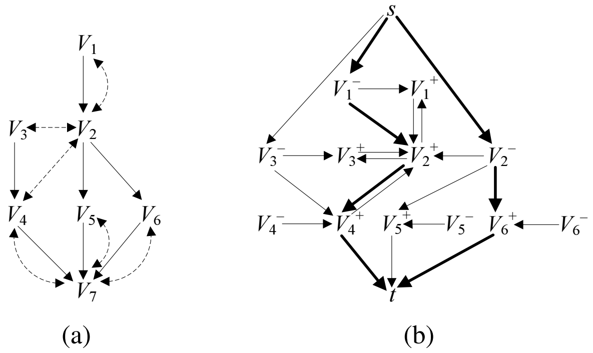
Note that if $Z_i = X_i$ then p_i degenerates into a node X_i.

Lemma 3 *The set of paths $p_1, \ldots, p_k$ satisfy the following properties*

- $Z_i \in NP_j$
- *$X_i \in Ne_j$, that is, $\alpha_{V_j X_i} \not\equiv 0$.*
- *Every intermediate node on p_i is a collider.*
- *Property 2 in Definition 2.*

As an example, for the model shown in Figure 7(a), the flow network for searching an accessory set relative to V_7 is given in Figure 7(b), which shows a maximum flow solution. The flow corresponds to paths $V_1 \to V_2 \leftrightarrow V_4$ and $V_2 \to V_6$ in the causal diagram.

From Lemma 3, we see that the set of variables $Z = \{Z_1, \ldots, Z_k\}$ satisfy all the properties of an accessory set except that Z_i or some intermediate nodes on the path p_i may not be ordered ahead of X_i. Next we show that we can always obtain a set of variables $X \subseteq Ne_j$ such that Z is an accessory set relative to X.



(a)

(b)

Figure 7: A SEM and corresponding flow network.

If there exist some nodes on p_i that is ordered after X_i, let the variable ordered the last on p_i be X_{j_i}.

Lemma 4 *1. $X_{j_i} \in Ne_j$, that is, $\alpha_{V_j X_{j_i}} \not\equiv 0$.*

2. X_{j_i} is not in $\{X_1, \ldots, X_k\}$.

For every path p_i that contains some nodes ordered after X_i, we replace X_i with X_{j_i} and the path p_i with $p_i[Z_i, X_{j_i}]$, the subpath of p_i between Z_i and X_{j_i}. We have that every node on $p_i[Z_i, X_{j_i}]$ is ordered ahead of X_{j_i}.

Theorem 2 *The set of variables $Z = \{Z_1, \ldots, Z_k\}$ is an accessory set relative to $\{X_{j_1}, \ldots, X_{j_k}\}$ and V_j, and Z is an accessory set with maximum size.*

7 Conclusion and Discussion

The identification problem has been a long standing problem in the applications of linear SEMs. In this paper, we provide a procedure for identifying individual path coefficients, adding a new sufficient identification criterion for dealing with the problem in practice. Our method is based on the partial regression equations, which appears to be a new promising research direction towards solving the identification problem.

The closest related work is the instrumental set method in [Brito and Pearl, 2002a]. Both of the instrumental set and the accessory set method provide sufficient conditions for identification, and it is not clear whether one method has more identification power than the other. The advantage of the accessory set method presented in this paper is that we provide an algorithm for identifying an accessory set with maximum size, while it is not clear how we can find an instrumental set systematically, to say nothing of an instrumental set with maximum size.

In this paper, we have treated α_{ij}'s as independent unknowns and obtained sufficient identification criterion. In certain class of models, α_{ij}'s are indeed independent parameters. We are investigating whether the criterion becomes also necessary in those situations.

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