Identifying Direct Causal Effects in Linear Models

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Abstract

This paper deals with the problem of identifying direct causal effects in recursive linear structural equation models. Using techniques developed for graphical causal models, we show that a model can be decomposed into a set of submodels such that the identification problem can be solved independently in each submodel. We provide a new identification method that identifies causal effects by solving a set of algebraic equations.

Introduction

Structural equation models (SEMs) have dominated causal reasoning in the social sciences and economics, in which interactions among variables are usually assumed to be linear (Duncan 1975; Bollen 1989). This paper deals with one fundamental problem in SEMs, accessing the strength of linear cause-effect relationships from a combination of observational data and model structures.

The problem has been under study for half a century, primarily by econometricians and social scientists, under the name "The Identification Problem"(Fisher 1966). Although many algebraic or graphical methods have been developed, the problem is still far from being solved. In other words, we do not have a necessary and sufficient criterion for deciding whether a causal effect can be computed from observed data. Most available methods are sufficient criteria which are applicable only when certain restricted conditions are met.

The contribution of this paper consists of two parts. First, we show how a model can be decomposed into a set of submodels such that the identification problem can be solved separately in each submodel. The technique is orthogonal to the available identification methods, and it is useful in practice because it is possible for an identification method which can not be applied to the full model to become applicable in smaller submodels. Second, we show a reduction of the identification problem into a problem of solving a set of algebraic equations. These equations provide an alternative to the classic Wright's rule (Wright 1934).

We begin with an introduction to SEMs and the identification problem, and give a brief review to previous work before presenting our new results.

$$X = \epsilon_x$$
$$W = aX + \epsilon_w$$
$$Z = bX + \epsilon_z$$
$$Y = cW + dZ + \epsilon_y$$
$$Cov(\epsilon_x, \epsilon_z) \neq 0$$
$$Cov(\epsilon_w, \epsilon_y) \neq 0$$

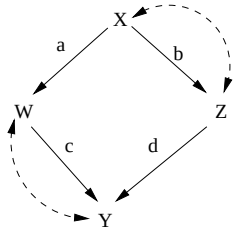


Figure 1: A linear SEM.

Linear SEMs and Identification

A linear SEM over a set of random variables $V = \{V_1, \ldots, V_n\}$ is given by a set of structural equations of the form

$$V_j = \sum_i c_{ji} V_i + \epsilon_j, \quad j = 1, \ldots, n, \tag{1}$$

where the summation is over the variables in V judged to be immediate causes of V_j. c_{ji}, called a *path coefficient*, quantifies the direct causal influence of V_i on V_j, and is also called a *direct effect*. ϵ_j's represent "error" terms and are assumed to have normal distribution. In this paper we consider recursive models and assume that the summation in Eq. (1) is for $i < j$, that is, $c_{ji} = 0$ for $i \geq j$. We denote the covariances between observed variables $\sigma_{ij} = Cov(V_i, V_j)$, and between error terms $\psi_{ij} = Cov(\epsilon_i, \epsilon_j)$. We denote the following matrices, $\Sigma = [\sigma_{ij}]$, $\Psi = [\psi_{ij}]$, and $C = [c_{ij}]$. Without loss of generality, the model is assumed to be standardized such that each variable V_j has zero mean and variance 1.

The structural assumptions encoded in a model are the zero path coefficient c_{ji}'s and zero error covariance ψ_{ij}'s. The model structure can be represented by a directed acyclic graph (DAG) G with (dashed) bidirected edges, called a *causal diagram* (or *path diagram*), as follows: the nodes of G are the variables $V_1, \ldots, V_n$; there is a directed edge from V_i to V_j in G if V_i appears in the structural equation for V_j, that is, $c_{ji} \neq 0$; there is a bidirected edge between V_i and V_j if the error terms ϵ_i and ϵ_j have non-zero correlation ($\psi_{ij} \neq 0$). Figure 1 shows a simple SEM and the corresponding causal diagram in which each directed edge is annotated by the corresponding path coefficient.

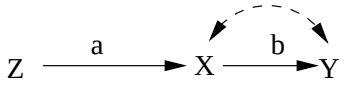


Figure 2: A typical instrumental variable

The parameters of the model are the non-zero entries in the matrices C and Ψ. Fixing the model structure and given parameters C and Ψ, the covariance matrix Σ is given by (see, for example, (Bollen 1989))

$$\Sigma = (I - C)^{-1}\Psi(I - C)^{T^{-1}}. \quad (2)$$

Conversely, in the identification problem, given the structure of a model, one attempts to solve for C in terms of the given covariance Σ. If Eq. (2) gives a unique solution to some path coefficient $c_{V_jV_i}$, independent of the (unobserved) error correlations Ψ, that path coefficient is said to be *identified*. In other words, the *identification problem* is that whether a path coefficient is determined uniquely from the covariance matrix Σ given the causal diagram. If every parameter of the model is identified, then *the model is identified*. Note that the identifiability conditions we seek involve the structure of the model alone, not particular numerical values of parameters, allowing for pathological exceptions.

Previous Work

Many methods have been developed for deciding whether a specific parameter or a model is identifiable. Traditional approaches are based on algebraic manipulation of the structural equations (Fisher 1966; Bekker, Merckens, & Wansbeek 1994; Rigdon 1995). Recently graphical approaches for identifying linear causal effects have been developed, and some sufficient graphical conditions were established (McDonald 1997; Pearl 1998; Spirtes *et al.* 1998; Pearl 2000; Tian 2004). The applications of these methods are limited in scope, and typically some special conditions have to be met for these methods to be applicable.

For example, the well-known instrumental variable (IV) method (Bowden & Turkington 1984) require search for variables (called *instruments*) that are uncorrelated with the error terms in specific equations. The graphical criterion for recognizing a variable Z as instrumental relative to a cause X and effect Y is described in (Pearl 2000). A typical configuration of the IV method is show in Fig. 2, in which Z serves as an instrument for identifying the causal effect b as

$$b = \sigma_{ZY}/\sigma_{ZX}. \quad (3)$$

One approach for the identification problem is to write Eq.(2) for each term σ_{ij} of Σ using Wright's method of path coefficients (Wright 1934). Wright's equations consist of equating the (standardized) covariance σ_{ij} with the sum of products of parameters (c_{ji}'s and ψ_{ji}'s) along all *unblocked paths* between V_i and V_j. A path is *unblocked* if there is no node X such that both edges connected to X in the path have an arrow at X ($\rightarrow X \leftarrow$). A path coefficient c_{ij} is identified if and only if Wright's equations give a unique solution to c_{ij}, independent of error correlations. For example, the Wright's equations for the model in Fig. 2 are

$$\begin{aligned} \sigma_{ZX} &= a \\ \sigma_{ZY} &= ab \\ \sigma_{XY} &= b + \psi_{XY} \end{aligned} \quad (4)$$

Recently, based on Wright's equations, sufficient graphical criteria for identification have been developed (Brito & Pearl 2002b; 2002a).

C-components and Identification

In recent years, causal reasoning with graphical causal models has been an active research area in the artificial intelligence community. The relation between linear SEMs and graphical causal models , in which typically no assumptions were made about the functional forms of how the variables interact with each other, is analyzed in (Spirtes *et al.* 1998; Pearl 1998; 2000). In this section, we will use the techniques developed in graphical causal models to derive a useful property for solving the identification problem.

Let PA_j denote the set of variables in V which appears in the equation (1) for V_j (PA_j are the set of parents of V_j in the causal diagram G.). Eq. (1) can be rewritten (in a nonparametric form) as[1]

$$v_j = f_j(pa_j, \epsilon_j) \quad j = 1, \ldots, n. \quad (5)$$

It can be shown that the causal model defined by Eq. (5) satisfies the Markov property that each variable V_j is conditionally independent of all its non-descendants in the causal diagram given pa_j and ϵ_j (Pearl 2000). These conditional independence assertions imply that the joint probability density function $p(v, \epsilon)$, where $V = \{V_1, \ldots, V_n\}$ and $\epsilon = \{\epsilon_1, \ldots, \epsilon_n\}$, can be decomposed (using the chain rule of probability calculus) into the product

$$p(v, \epsilon) = \prod_i p(v_i|pa_i, \epsilon_i)p(\epsilon) \quad (6)$$

Then the joint over observed variables is given by

$$p(v) = \int_\epsilon \prod_i p(v_i|pa_i, \epsilon_i)p(\epsilon)d\epsilon \quad (7)$$

For linear SEMs, $p(v) \sim N(0, \Sigma)$ is a normal density with covariance matrix Σ, $p(\epsilon) \sim N(0, \Psi)$, and each $p(v_i|pa_i, \epsilon_i)$ is a Dirac delta function

$$p(v_i|pa_i, \epsilon_i) = \delta(v_i - \sum_{V_l \in PA_i} c_{il}v_l - \epsilon_i) \quad (8)$$

Using the property of the Dirac delta function, Eq. (7) can be rewritten as

$$p(v) = p(\epsilon)|_{\epsilon_i = v_i - \sum_{V_l \in PA_i} c_{il}v_l} \quad (9)$$

Therefore Eq. (7) can be taken as an equation for Σ in terms of C and Ψ which should be equivalent to Eq. (2).

[1]We use uppercase letters to represent variables or sets of variables, and use corresponding lowercase letters to represent their values (instantiations).

The integration of products in Eq. (7) can in fact be factorized into a product of integrations depending on the structure of the causal diagram (Tian & Pearl 2002). Let a path composed entirely of bidirected edges be called a *bidirected path*. The set of variables V in the causal diagram can be partitioned into disjoint groups by assigning two variables to the same group if and only if they are connected by a bidirected path. Assume that V is thus partitioned into k groups $S_1, \ldots, S_k$, each called a *c-component* of V in G or a c-component of G. Let N_j be the set of ϵ variables that correspond to those variables in S_j (ϵ_i corresponds to V_i). Then the sets $N_1, \ldots, N_k$ form a partition of ϵ. We have that if $\epsilon_i \in N_m$ and $\epsilon_j \in N_l$ belong to different groups then they are uncorrelated, that is, $\psi_{ij} = 0$ for $m \neq l$. Therefore we have

$$p(\epsilon) = \prod_{j=1}^{k} p(n_j) \tag{10}$$

Define, for $j = 1, \ldots, k$,

$$Q_j = \int_{n_j} \prod_{\{i|V_i \in S_j\}} p(v_i|pa_i, \epsilon_i) p(n_j) dn_j. \tag{11}$$

From Eq. (10), $P(v)$ in Eq. (7) can be factorized into a product of Q_j's:

$$P(v) = \prod_{j=1}^{k} Q_j. \tag{12}$$

For example, in the model of Figure 1, V is partitioned into the c-components $\{X, Z\}$ and $\{W, Y\}$, and if we define

$$Q_1 = \int p(x|\epsilon_x) p(z|x, \epsilon_z) p(\epsilon_x, \epsilon_z) d\epsilon_x d\epsilon_z, \tag{13}$$

and

$$Q_2 = \int p(w|x, \epsilon_w) p(y|w, z, \epsilon_y) p(\epsilon_w, \epsilon_y) d\epsilon_w d\epsilon_y, \tag{14}$$

then we have

$$p(x, w, y, z) = Q_1 Q_2 \tag{15}$$

The importance of this factorization stems from that each Q_j is computable from $p(v)$, as shown in the following lemma.

Lemma 1 *(Tian & Pearl 2002) Let $V^{(i)} = \{V_1, \ldots, V_i\}$, $i = 1, \ldots, n$, and $V^{(0)} = \emptyset$. Q_j can be computed as*

$$Q_j = \prod_{\{i|V_i \in S_j\}} p(v_i|v^{(i-1)}), \quad j = 1, \ldots, k. \tag{16}$$

For example, in the model of Figure 1, Q_1 defined in Eq. (13) is given by

$$Q_1 = p(x)p(z|x) \tag{17}$$

and Q_2 defined in Eq. (14) is given by

$$Q_2 = p(w|x)p(y|w, z, x). \tag{18}$$

From the definition of Q_j in Eq. (11), we rewrite Eq. (16) as

$$\prod_{\{i|V_i \in S_j\}} p(v_i|v^{(i-1)}) = \int_{n_j} \prod_{\{i|V_i \in S_j\}} p(v_i|pa_i, \epsilon_i) p(n_j) dn_j \tag{19}$$

By Eq. (8) and the property of the Dirac delta function, Eq. (19) can be rewritten as

$$\prod_{\{i|V_i \in S_j\}} p(v_i|v^{(i-1)}) = p(n_j)|_{\epsilon_i = v_i - \sum_{V_l \in PA_i} c_{il} v_l}. \tag{20}$$

Therefore, Eq. (19) represents a set of equations for Σ in terms of those parameters (c_{ij}'s and ψ_{ij}'s) that appear in the structural equations for variables in S_j. We obtain that the set of equations in Eq. (2) can be divided into k independent sets of equations represented by Eq. (19), each only involving parameters corresponding to the variables in a c-component. And we get the following lemma.

Lemma 2 *Let a variable V_i be in a c-component S_j. Eq. (2) gives a unique solution to a path coefficient c_{im} iff the set of equations represented by Eq. (19) gives a unique solution to c_{im}.*

We now transform the problem represented by Eq. (19) back into the form of a set of linear structural equations. Let $Pa(S)$ denote the union of a set S and the set of parents of S, that is, $Pa(S) = S \cup (\cup_{V_i \in S} PA_i)$. Let $Parent(S) = Pa(S) \setminus S$. Multiply both sides of Eq. (19) by $\prod_{\{l|V_l \in Parent(S_j)\}} p(v_l)$ which is equal to $\prod_{\{l|V_l \in Parent(S_j)\}} \int p(v_l|\epsilon_l) p(\epsilon_l) d\epsilon_l$

$$\begin{aligned}
&\prod_{\{i|V_i \in S_j\}} p(v_i|v^{(i-1)}) \prod_{\{l|V_l \in Parent(S_j)\}} p(v_l) \\
&= \int \prod_{\{i|V_i \in S_j\}} p(v_i|pa_i, \epsilon_i) \Big(\prod_{\{l|V_l \in Parent(S_j)\}} p(v_l|\epsilon_l) p(\epsilon_l) \Big) \\
&\quad \cdot p(n_j) dn_j \prod_{\{l|V_l \in Parent(S_j)\}} d\epsilon_l
\end{aligned} \tag{21}$$

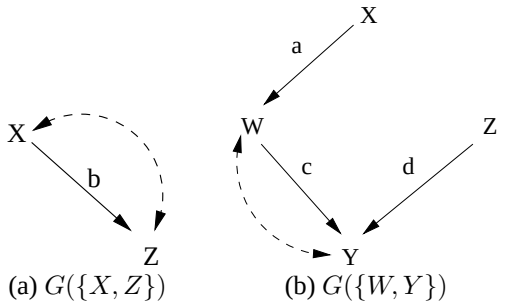
Define the left side of Eq. (21) as a distribution p' over the set of variables $Pa(S_j)$ as[2]

$$p'(pa(S_j)) = \prod_{\{i|V_i \in S_j\}} p(v_i|v^{(i-1)}) \prod_{\{l|V_l \in Parent(S_j)\}} p(v_l) \tag{22}$$

Comparing the right hand side of Eq. (21) with that of Eq. (7), we conclude that Eq. (21), and therefore Eq. (19), represents a set of structural equations over the set of variables $Pa(S_j)$ given by

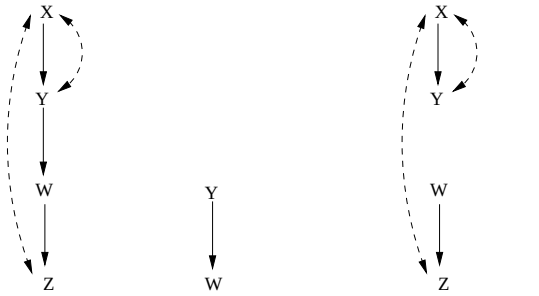
$$\begin{aligned}
V_l &= \epsilon_l, & V_l \in Parent(S_j) \\
V_i &= \sum_{V_m \in PA_i} c_{im} V_m + \epsilon_i, & V_i \in S_j
\end{aligned} \tag{23}$$

[2](Tian & Pearl 2002) shows $\prod_{\{i|V_i \in S_j\}} p(v_i|v^{(i-1)})$ is indeed only a function of $Pa(S_j)$.



(a) $G(\{X, Z\})$ (b) $G(\{W, Y\})$

Figure 3: The SEM in Fig. 1 is decomposed into two submodels



(a) G (b) $G(\{W\})$ (c) $G(\{X, Y, Z\})$

Figure 4: A SEM model

such that $Pa(S_j)$ is distributed as $p'(pa(S_j))$ given by Eq. (22). The covariance matrix over $Pa(S_j)$, denoted by $\Sigma'_j = [\sigma'_{ij}]$, can be computed from the density $p'(pa(S_j))$ as

$$\sigma'_{ij} = \int v_i v_j p'(pa(S_j)) dpa(S_j). \tag{24}$$

The preceding analysis leads to the following lemma.

Lemma 3 *A path coefficient c_{im} is identifiable in the SEM given by Eq. (23) iff the set of equations represented by Eq. (19) gives a unique solution to c_{im}.*

Given a SEM M with causal diagram G consisting of c-components $S_1, \ldots, S_k$, the SEM given by Eq. (23) with corresponding Σ'_j will be denoted by $M(S_j)$, and the causal diagram of $M(S_j)$ will be denoted by $G(S_j)$. $G(S_j)$ can be obtained from the subgraph of G composed only of variables in $Pa(S_j)$ by deleting all the arrows pointing toward a variable in $Parent(S_j)$, which makes sure that each variable in $Parent(S_j)$ is a root node in $G(S_j)$.

Lemmas 2 and 3 lead to the following two theorems.

Theorem 1 *Let a variable V_i be in a c-component S_j in a SEM M. A path coefficient c_{im} is identifiable iff it is identifiable in the model $M(S_j)$.*

Proof: c_{im} is identifiable iff Eq. (2) gives a unique solution to c_{im}, then by Lemma 2, iff the set of equations represented by Eq. (19) gives a unique solution to c_{im}, finally by Lemma 3, iff c_{im} is identifiable in the model $M(S_j)$. □

Theorem 2 *Let V be partitioned into c-components $S_1, \ldots, S_k$ in a SEM M. M is identifiable iff each model $M(S_j)$, $j = 1, \ldots, k$, is identifiable.*

Proof: M is identifiable iff each path coefficient is identifiable, and by Theorem 1, iff each model $M(S_j)$ is identifiable. □

For example, the identifiability problem for the model shown in Fig. 1 is reduced to that in two simpler models shown in Fig. 3(a) and (b), with their density functions given by (see Eqs. (17), (18), and (22))

$$p'(x, z) = p(x)p(z|x) = p(x, z) \tag{25}$$

and

$$p'(x, z, w, y) = p(w|x)p(y|w, z, x)p(x)p(z) \tag{26}$$

respectively.

The results in Theorems 1 and 2 are orthogonal to the existing identification methods. We can apply these two theorems to decompose the model into some simpler models before attempting to use available identification method. The decomposition will often be useful in practice because we do not have a necessary and sufficient criterion or procedure for deciding the identifiability of either path coefficients or the whole model. Available methods are typically sufficient criteria that are only applicable when some special conditions are met. It is possible that an identification method that can not be applied to the original model becomes applicable in the reduced models.

For example, the IV method can not be directly applied to identify the path coefficient c in the model shown in Fig. 1, because X is correlated with Z which appears in the structural equation for Y. However Theorem 1 says that c is identifiable if and only if it is identifiable in the model shown in Fig. 3(b). Clearly X can serve as an instrument for identifying c in the model in Fig. 3(b) (see Fig. 2), and we conclude that c is identified as $c = \sigma'_{XY}/\sigma'_{XW}$ where σ'_{XY} and σ'_{XW} can be computed from Eq. (26).

As another example, we consider the model M in Fig. 4(a) which appears in (Brito & Pearl 2002b) as an example to show that their identification criterion is not complete, that is, the model M is identifiable but their criterion is not applicable. Applying Theorem 2, we obtain that M is identifiable if and only if both the models in Fig. 4(b) and (c) are identifiable. Now the criterion in (Brito & Pearl 2002b) can actually be applied to both models to show that they are identified.

Identification by Regression

In this section, we propose a method for identification that transforms each of the structural equations into a *regression* equation. A linear equation

$$V_i = \sum_{j=1}^{m} c_j V_{i_j} + \epsilon \tag{27}$$

is a regression if and only if ϵ is uncorrelated with each V_{i_j},

$$Cov(V_{i_j}, \epsilon) = 0, \quad j = 1, \ldots, m. \tag{28}$$

When Eq. (27) is a regression, each coefficient c_j is identifiable. Let $S = \{V_{i_1}, \ldots, V_{i_m}\}$ and $S_j = S \setminus \{V_{i_j}\}$. Then we have

$$c_j = \beta_{ii_j.S_j} \tag{29}$$

where $\beta_{ii_j.S_j}$ denotes the *partial regression coefficient* and represents the coefficient of V_{i_j} in the linear regression of V_i on S. (Note that the order of the subscripts in $\beta_{ij.S}$ is essential.) Partial regression coefficients can be expressed in terms of covariance matrices as follows (Cramer 1946):

$$\beta_{ij.S} = \frac{\Sigma_{V_iV_j} - \Sigma_{V_iS}^T \Sigma_{SS}^{-1} \Sigma_{V_jS}}{\Sigma_{V_jV_j} - \Sigma_{V_jS}^T \Sigma_{SS}^{-1} \Sigma_{V_jS}}, \tag{30}$$

where Σ_{SS} etc. represent covariance matrices over corresponding variables.

We introduce our idea by studying the model shown in Fig. 2 whose structural equations are

$$Z = \epsilon_z \tag{31}$$
$$X = aZ + \epsilon_x \tag{32}$$
$$Y = bX + \epsilon_y \tag{33}$$
$$\psi_{yx} = Cov(\epsilon_y, \epsilon_x) \neq 0$$

We assert that Eq. (32) is a regression since $\psi_{xz} = 0$. Therefore we have

$$\beta_{XZ} = a, \tag{34}$$

and Eq. (32) can be rewritten as

$$X = \beta_{XZ} Z + \epsilon_x \tag{35}$$

Eq. (33) is not a regression since $\psi_{yx} \neq 0$. If we define

$$\epsilon'_y = \epsilon_y - \frac{\psi_{xy}}{\psi_{xx}} \epsilon_x, \tag{36}$$

then we have

$$Cov(\epsilon'_y, \epsilon_x) = Cov(\epsilon'_y, \epsilon_z) = 0 \tag{37}$$

Obtaining the expression for ϵ_y from Eq. (36) and substituting it into Eq. (33), we get

$$Y = bX + \frac{\psi_{xy}}{\psi_{xx}} \epsilon_x + \epsilon'_y \tag{38}$$

Substituting into Eq. (38) the expression for ϵ_x obtained from Eq. (35)

$$Y = (b + \frac{\psi_{xy}}{\psi_{xx}})X - \beta_{XZ} \frac{\psi_{xy}}{\psi_{xx}} Z + \epsilon'_y \tag{39}$$

From Eq. (37), we conclude that Eq. (39) is a regression, and we obtain

$$\beta_{YX.Z} = b + \frac{\psi_{xy}}{\psi_{xx}} \tag{40}$$

$$\beta_{YZ.X} = -\beta_{XZ} \frac{\psi_{xy}}{\psi_{xx}} \tag{41}$$

Therefore we have transformed the set of structural equations in Eqs. (31)-(33) into a set of regression equations, and obtained expressions for partial regression coefficients in terms of the path coefficients and error covariances as given in Eqs. (34), (40), and (41). Solving these three equations leads to unique solutions for a and b, and we conclude that a and b are identifiable.

Next we show that we can transform a general SEM given in Eq. (1) into a set of regression equations. First we will "orthogonalize" the set of error terms to obtain a new set of error terms $\{\epsilon'_1, \ldots, \epsilon'_n\}$ that are mutually orthogonal in the sense that

$$Cov(\epsilon'_i, \epsilon'_j) = 0, \text{ for } i \neq j. \tag{42}$$

We will use the Gram-Schmidt orthogonalization process. The process proceeds recursively as follows. We set

$$\epsilon'_1 = \epsilon_1 \tag{43}$$

For $j = 2, \ldots, n$, we set

$$\epsilon'_j = \epsilon_j - \sum_{k=1}^{j-1} \alpha_{jk} \epsilon'_k \tag{44}$$

in which

$$\alpha_{jk} = \frac{Cov(\epsilon_j, \epsilon'_k)}{Cov(\epsilon'_k, \epsilon'_k)}. \tag{45}$$

Then Eq. (42) is guaranteed to hold.

Then we will transform the set of structural equations in (1) into regression equations recursively as follows. For $j = 1$, substitute Eq. (43) into Eq. (1):

$$V_1 = \epsilon'_1 \tag{46}$$

For $j = 2$, substitute the expression for ϵ_2 obtained from Eq. (44) into Eq. (1):

$$V_2 = c_{21} V_1 + \alpha_{21} \epsilon'_1 + \epsilon'_2 \tag{47}$$

Substitute the expression for ϵ'_1 obtained from Eq. (46) into Eq. (47):

$$V_2 = (c_{21} + \alpha_{21}) V_1 + \epsilon'_2 \tag{48}$$

Since $Cov(\epsilon'_2, \epsilon'_1) = 0$, Eq. (48) is a regression equation. We have

$$\beta_{21} = c_{21} + \alpha_{21}, \tag{49}$$

and Eq. (48) can be rewritten as

$$V_2 = \beta_{21} V_1 + \epsilon'_2 \tag{50}$$

For $j = 3, \ldots, n$, we substitute the expression for ϵ_j obtained from Eq. (44) into Eq. (1)

$$V_j = \sum_{i<j} c_{ji} V_i + \sum_{k=1}^{j-1} \alpha_{jk} \epsilon'_k + \epsilon'_j. \tag{51}$$

If for $k = 1, \ldots, j-1$, each of the Eq. (1) for V_k can be rewritten as a regression equation

$$V_k = \sum_{i=1}^{k-1} \beta_{ki.S_{ki}} V_i + \epsilon'_k, \tag{52}$$

where

$$S_{ki} = \{V_1, \ldots, V_{k-1}\} \setminus \{V_i\}, \tag{53}$$

then substituting the expression for ϵ'_k's obtained from Eq. (52) into Eq. (51)

$$\begin{aligned} V_j &= \sum_{i<j} c_{ji} V_i + \sum_{k=1}^{j-1} \alpha_{jk} (V_k - \sum_{i=1}^{k-1} \beta_{ki.S_{ki}} V_i) + \epsilon'_j \\ &= \sum_{k=1}^{j-1} (c_{jk} + \alpha_{jk} - \sum_{k+1 \le l \le j-1} \beta_{lk.S_{lk}} \alpha_{jl}) V_k + \epsilon'_j \end{aligned} \tag{54}$$

From Eq. (42) and Eq. (52) it is easy to show that

$$Cov(\epsilon'_j, V_k) = 0 \text{ for } k = 1, \ldots, j-1. \tag{55}$$

Therefore Eq. (54) is also a regression equation. We have in fact proved, by induction, that Eq. (54) is a regression equation for $j = 2, \ldots, n$, and we obtain

$$\begin{aligned} \beta_{jk.S_{jk}} = c_{jk} + \alpha_{jk} - \sum_{k+1 \le l \le j-1} \beta_{lk.S_{lk}} \alpha_{jl}, \\ j = 2, \ldots, n, \ k = 1, \ldots, j-1, \end{aligned} \tag{56}$$

where S_{ij}'s are defined in Eq. (53).

Eq. (56) expresses a partial regression coefficient, for each pair of variables, in terms of the parameters (path coefficients and error covariances) of the model. Given the model structure (represented by zero path coefficients and zero error correlations), some of the c_{jk}'s and α_{jk}'s will be set to zero in Eq. (56), and we can solve the identifiability problem by solving Eq. (56) for c_{jk}'s in terms of the partial regression coefficients. This provides an alternative to the Wright's equation method, and we will refer to Eq. (56) as the *partial regression equations*. A path coefficient c_{ij} is identified if and only if the set of partial regression equations give a unique solution to c_{ij}, independent of error correlations. The partial regression equations are linear with respect to path coefficient c_{jk}'s and α_{jk}'s (although not linear with respect to ψ_{ij}'s), while Wright's equations are nonlinear with respect to c_{jk}'s. As a consequence, the partial regression equations may have some advantages over Wright's equations when we look for computer programs that can identify path coefficients automatically by solving these algebraic equations.

As an example, the partial regression equations for the model shown in Fig. 1 are given by

$$\beta_{WX} = a \tag{57}$$

$$\beta_{ZW.X} = 0 \tag{58}$$

$$\beta_{ZX.W} = b + \alpha_{ZX} \tag{59}$$

$$\beta_{YZ.WX} = d \tag{60}$$

$$\beta_{YW.XZ} = c + \alpha_{YW} \tag{61}$$

$$\beta_{YX.WZ} = -\beta_{WX} \alpha_{YW} \tag{62}$$

which happens to be linear with respect to all the parameters. It is not difficult to solve these equations to obtain that the path coefficients a, d, and c are identified. On the other hand, the Wright's equations for this model are nonlinear and would be difficult to solve.

Conclusion

Using graphical model techniques, we show that a SEM can be decomposed into some submodels such that the identification problem can be solved independently in each submodel. The decomposition can serve as a preprocessing step before attempting to apply existing identification methods.

We derive an expression for the partial regression coefficient $\beta_{ji.S_{ji}}$, for each pair of variables V_i and V_j, in terms of the model parameters, while the classic Wright's rule expresses each covariance σ_{ij} in terms of the model parameters. This provides a new principled method for solving the identifiability problem since the identifiability of a parameter or the model can be determined by solving the resulting algebraic equations for path coefficients. The method may have advantages over Wright's method since the set of partial regression equations are linear with respect to path coefficients. For models with a few variables, the set of partial regression equations are typically much easier to solve by human experts than Wright's equations so as to obtain necessary and sufficient identification results. However, the set of partial regression equations may be nonlinear with respect to error covariances, and (as Wright's method) therefore cannot be directly used as identification criterion, rather it provides a new tool for deriving identification criteria. We are currently working on deriving identification criteria using the set of partial regression equations.

Wright's equations can be written down by inspecting the causal diagram. We are still investigating how the partial regression equations are related to the paths in the causal diagram.

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