Ideals with Large(r) Projective Dimension and Stillman's Question

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Notation

K a field

$S = K[X_1, X_2, \ldots, X_n]$

$S = \bigoplus_{i=0}^{\infty} S_i$ is graded

$S(-d)_i = S_{i-d}$ = rank one free module with generator in degree d

$I = (f_1, \ldots, f_t) \subset S$ a homogeneous ideal

(i.e. each f_j is in some S_i)

Graded Free Resolutions

Compute a graded free resolution of S/I:

$$0 \leftarrow S/I \leftarrow S \leftarrow \bigoplus_j S(-d_j)^{\beta_{1j}} \leftarrow \bigoplus_j S(-d_j)^{\beta_{2j}} \leftarrow \cdots \leftarrow \bigoplus_j S(-d_j)^{\beta_{pj}} \leftarrow 0$$

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$$\text{Projective Dimension of } S/I = \mathrm{pd}(S/I) = \max\{i \mid \beta_{ij} \neq 0\}$$

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$$\text{Projective Dimension of } S/I = \mathrm{pd}(S/I) = \max\{i \mid \beta_{ij} \neq 0\}$$

$$\text{Regularity of } S/I = \mathrm{reg}(S/I) = \max\{j - i \mid \beta_{ij} \neq 0\}$$

Betti Diagrams

Record the Betti numbers β_{ij} in a matrix called the Betti table:

	0	1	2	$\cdots$	i	$\cdots$
0:	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	$\cdots$	$\beta_{i,i}$	$\cdots$
1:	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	$\cdots$	$\beta_{i,i+1}$	$\cdots$
2:	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\cdots$	$\beta_{i,i+2}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
j:	$\beta_{0,j}$	$\beta_{1,j+1}$	$\beta_{2,j+2}$	$\cdots$	$\beta_{i,i+j}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	

$\mathrm{pd}(S/I)$ = last nonzero column in Betti table.
$\mathrm{reg}(S/I)$ = last nonzero row in Betti table.

Burch-Kohn Theorem (Polynomial Ring Case)

Theorem

For any $n \in \mathbb{N}$, there is a three-generated ideal $I = (f, g, h)$ in a polynomial ring $S = K[x_1, \ldots, x_{2n}]$ with $\mathrm{pd}(S/I) = n$.

Remark

Engheta computed the degrees of the 3 generators:
$n - 2, n - 2, 2n - 2$. (linear growth with respect to n)

Stillman's Question

Question (Stillman)

Is there a bound, independent of n, on the projective dimension of ideals in $S = K[X_1, \ldots, X_n]$ which are generated by N homogeneous polynomials of given degrees $d_1, \ldots, d_N$?

Remark

Hilbert Syzygy Theorem guarantees $\mathrm{pd}(S/I) \leq n$, but we seek a bound independent of n.

Stillman's Question

Known Cases:

1. $I = (m_1, \ldots, m_N)$ is a monomial ideal. Then $\mathrm{pd}(S/I) \leq N$. (Taylor resolution)
2. $I = (f, g, h)$ with $d_1 = d_2 = d_3 = 2$. Then $\mathrm{pd}(S/I) \leq 4$. (Eisenbud-Huneke) e.g.: $I = (w^2, x^2, wy + xz)$ satisfies $\mathrm{pd}(S/I) = 4$. So bound is tight.
3. $I = (f, g, h)$ with $d_1 = d_2 = d_3 = 3$. Then $\mathrm{pd}(S/I) \leq 36$. (Engheta)
   Largest known example: $I = (x^3, y^3, x^2a + xyb + y^2c)$ satisfies $\mathrm{pd}(S/I) = 5$. Bound is likely not tight.
4. $I = (f_1, \ldots, f_N)$ with $d_1 = \cdots = d_N = 2$. Then $\mathrm{pd}(S/I) \leq 2N^{2N}$. (Ananyan-Hochster)

Zhang's Question

Question (Zhang)

Given a homogeneous ideal I in $S = K[X_1, \ldots, X_n]$ with N generators of degrees $d_1, \ldots, d_N$, is

$$\mathrm{pd}(S/I) \leq \sum_{i=1}^{N} d_i?$$

The following construction shows that this bound is too small.

Definition

Fix integers m, n, d such that $m \geq 1$, $n \geq 0$ and $d \geq 2$.
Order the $M_{m,d-1} = \frac{(m+d-2)!}{(m-1)!(d-1)!}$ monomials of degree $d-1$ over the variables $X_1, \ldots, X_m$.
$Z_i = i^{th}$ such monomial
$p = M_{m,d-1}$
$S = K[X_1, \ldots, X_m, Y_{1,1}, \ldots, Y_{p,n}]$

$$I_{m,n,d} = \left(X_1^d, \ldots, X_m^d, \sum_{j=1}^{p} Z_j Y_{j,1}, \ldots, \sum_{j=1}^{p} Z_j Y_{j,n} \right)$$

$m + n$ degree-d generators

Theorem (-)

$$\mathrm{pd}(R/I_{m,n,d}) = m + np = m + n\frac{(m+d-2)!}{(m-1)!(d-1)!}.$$

Proof Sketch

Theorem (Homogeneous Auslander-Buchsbaum)

For any graded S-module M,

$$\operatorname{depth} M + \operatorname{pd} M = \operatorname{depth} S$$

where $\operatorname{depth} M$ = length of maximal regular sequence on M in the graded maximal ideal.

$\operatorname{pd}(S/I_{m,n,d}) = \dim S \Leftrightarrow \operatorname{depth}(S/I_{m,n,d}) = 0$
$\Leftrightarrow \exists s \in S - I$ with $Xs \in I$ for every variable X of S.
Check that $X_1^{d-1} X_2^{d-1} \cdots X_m^{d-1}$ works.

Example: $I = I_{3,4,2}$

$S = K[X_1, \ldots, X_3, Y_{1,1}, \ldots, Y_{3,4}]$
$I = (X_1^2, X_2^2, X_3^2, X_1 Y_{1,1} + X_2 Y_{2,1} + X_3 Y_{3,1}, X_1 Y_{1,2} + X_2 Y_{2,2} + X_3 Y_{3,2}, X_1 Y_{1,3} + X_2 Y_{2,3} + X_3 Y_{3,3}, X_1 Y_{1,4} + X_2 Y_{2,4} + X_3 Y_{3,4})$
7 quadratic generators
$\mathrm{pd}(S/I) = \#\text{variables} = 15$

Three-Generated Case: $I = I_{2,1,d}$

$S = K[X_1, X_2, Y_1, \ldots, Y_d]$
$I = (X_1^d, X_2^d, X_1^{d-1}Y_1 + X_1^{d-2}X_2Y_2 + X_1^{d-3}X_2^2Y_3 + \cdots + X_2^{d-1}Y_d)$
Three generators in degree d.
$\mathrm{pd}(S/I) = d + 2$

Question

Can we find a three-generated ideal with $\mathrm{pd}(S/I) > 3d$*?*

Definition

Fix integers $g \geq 2$, $m_1, \ldots, m_n$ with $m_n \geq 0, m_{n-1} \geq 1$, and $m_i \geq 2$ for $i < n$.
We define an ideal with $g+1$ generators of degree $d = 1 + \sum_i m_i$.
Notation:

$$\mathcal{A}_k = \left\{ (a_{j,k'}) \;\middle|\; \begin{array}{l} \text{Only first k columns are nonzero} \\ \sum_{j=1}^{g} a_{j,i} = m_i \text{ with at least 2 nonzero en-} \\ \text{tries for } i \leq k \\ \text{(or just } \sum_{j=1}^{g} a_{j,n} = m_i \text{ when } i = n \leq k.) \end{array} \right\}$$

Definition

$$S = K[\mathrm{X}, y_{\mathrm{A}} \mid \mathrm{X} = (x_{j,k}), \mathrm{A} \in \mathcal{A}_n],$$

$$I_{g,(m_1,\dots,m_n)} = (x_{1,1}^d, \dots, x_{g,1}^d, f),$$

where

$$f = \sum_{k=1}^{n-1} \sum_{\mathrm{A} \in \mathcal{A}_{k-1}} \sum_{j=1}^{g} \mathrm{X}^{\mathrm{A}} x_{j,k}^{m_k} x_{j,k+1}^{d_{k+1}} + \sum_{\mathrm{B} \in \mathcal{A}_n} \mathrm{X}^{\mathrm{B}} y_{\mathrm{B}}.$$

$(d_k = m_k + \cdots + m_n + 1)$

Theorem (Beder,Nunez,Seceleanu,Snapp,Stone,-)

Using the notation above with $I = I_{g,(m_1,\ldots,m_n)}$, $\operatorname{depth}(R/I) = 0$.

Proof Sketch: Set $\mathrm{T} = (t_{j,k})$ and $t_{j,k} = d_k - 1$.
Set $\mathrm{S} = \mathrm{X}^{\mathrm{T}}$.
Show $\mathrm{S} \in (I : \mathfrak{m}) - I$.

Corollary

$$\operatorname{pd}(R/I) = \prod_{i=1}^{n-1} \left(\frac{(m_i + g - 1)!}{(g-1)!(m_i)!} - g \right) \left(\frac{(m_n + g - 1)!}{(g-1)!(m_n)!} \right) + gn.$$

Proof: Count the variables. $g \times n$ X variables and $|\mathcal{A}_n|$ Y variables.

Example: $I = I_{2,(3,1)}$

$$S = K\left[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, y_{\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right)}, y_{\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)}, y_{\left(\begin{smallmatrix} 1 & 1 \\ 2 & 0 \end{smallmatrix}\right)}, y_{\left(\begin{smallmatrix} 2 & 0 \\ 1 & 1 \end{smallmatrix}\right)}\right]$$

$$I = (x_{1,1}^5, x_{2,1}^5, x_{1,1}^3x_{1,2}^2 + x_{2,1}^3x_{2,2}^2 + x_{1,1}^2x_{1,2}x_{2,1}y_{\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right)} + x_{1,1}x_{2,1}^2x_{2,2}y_{\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)} + x_{1,1}x_{1,2}x_{2,1}^2y_{\left(\begin{smallmatrix} 1 & 1 \\ 2 & 0 \end{smallmatrix}\right)} + x_{1,1}^2x_{2,1}x_{2,2}y_{\left(\begin{smallmatrix} 2 & 0 \\ 1 & 1 \end{smallmatrix}\right)})$$

By previous Theorem,

$$\operatorname{pd}(S/I) = \left(\frac{(3+2-1)!}{(2-1)!(3)!} - 2\right)\left(\frac{(1+2-1)!}{(2-1)!(1)!}\right) + 2 \cdot 2 = 8.$$

Example: $I = I_{2,(3,1)}$

	0	1	2	3	4	5	6	7	8
total:	1	3	53	184	287	248	124	34	4
0:	1	-	-	-	-	-	-	-	-
1:	-	-	-	-	-	-	-	-	-
2:	-	-	-	-	-	-	-	-	-
3:	-	-	-	-	-	-	-	-	-
4:	-	3	-	-	-	-	-	-	-
5:	-	-	-	-	-	-	-	-	-
6:	-	-	-	-	-	-	-	-	-
7:	-	-	-	-	-	-	-	-	-
8:	-	-	3	-	-	-	-	-	-
9:	-	-	3	4	-	-	-	-	-
10:	-	-	13	46	68	56	28	8	1
11:	-	-	33	132	218	192	96	26	3
12:	-	-	1	2	1	-	-	-	-

Corollary

Over any field K and for any positive integer p, there exists an ideal I in a polynomial ring S over K with three homogeneous generators in degree p^2 such that $\mathrm{pd}(R/I) \geq p^{p-1}$.

Proof:

$$I = I_{2,(\underbrace{p+1,\ldots,p+1}_{p-1 \text{ times}},0)}.$$

Corollary

Over any field K and for any positive integer p, there exists an ideal I in a polynomial ring R over K with $2p+1$ homogeneous generators in degree $2p+1$ such that $\mathrm{pd}(R/I) \geq p^{2p}$.

Proof:

$$I_{2p,\underbrace{(2,2,2,\dots,2)}_{p \text{ times}}}.$$

Note: The first family $I_{m,1,d}$ is a subfamily of the new family.

$$I_{m,1,d} = I_{m,(d-1)}$$

(up to a relabeling of the variables)

Let

$$C_d = (w^d, x^d, wy^{d-1} + xz^{d-1}) \subset S = K[w, x, y, z]$$

Caviglia showed $\mathrm{reg}(S/C_d) = d^2 - 1$.
C_d is also a subfamily the new family.

$$C_d = I_{2,(1,d-2)}$$

(up to a relabeling of the variables)

Question

What is $\mathrm{reg}(S/I_{g,(m_1,\ldots,m_n)})$*?*

Caviglia's family has quadratic regularity growth relative to degree of the 3 gens. Some of our ideals have larger regularity...

Example: $I = I_{2,(2,1,2)}$

$$S = K\left[x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}\right]$$

$$I = (x_{1,1}^6, x_{2,1}^6, x_{1,1}^2x_{1,2}^4 + x_{2,1}^2x_{2,2}^4 + x_{1,1}x_{2,1}x_{1,2}x_{1,3}^3 + x_{1,1}x_{2,1}x_{2,2}x_{2,3}^3)$$

I has three degree 6 generators and $\mathrm{reg}(S/I) = 41$.

Stillman's Question - Regularity Version

Question (Stillman)

Is there a bound, independent of n, on the **regularity** *of ideals in $S = K[X_1, \ldots, X_n]$ which are generated by N homogeneous polynomials of given degrees $d_1, \ldots, d_N$?*

Caviglia proved: This question $\Leftrightarrow$ PD question.
(See Bahman Engheta's thesis.)

Questions

Note: Socle elements grow linearly with degree.
$\Rightarrow \operatorname{reg}(S/I_{2,(d-1)}) \geq 2d-2$. Caviglia showed
$\operatorname{reg}(S/I_{2,(1,d-2)}) \geq d^2-1$.

Question

Does $\operatorname{reg}(S/I_{2,(2,1,d)})$ *exhibit cubic growth?*

Question

Does $\operatorname{reg}(S/I_{2,(2,2,2,2,\dots,2)})$ *exhibit exponential growth?*

Thank you!

References:

- T. Ananyan, M. Hochster, "Ideals Generated by Quadratic Polynomials." arXiv:1106.0839v1 [math.AC].
- J. Caviglia, "Koszul Algebras, Castelnuovo-Mumford Regularity, and Generic Initial Ideals." PhD Thesis. U. of Kansas. 2005
- B. Engheta, "Bounding Projective Dimension." PhD Thesis. U. of Kansas. 2005
- J. McCullough, "A Family of Ideals with Few Generators in Low Degree and Large Projective Dimension." Proceedings of the AMS. Volume 139 (2011), No. 6. pp. 2017-2023.
- J Beder, J McCullough, L Nunez, A Seceleanu, B Snapp, B Stone, "Ideals with Larger Projective Dimension and Regularity." Journal of Symbolic Computation 46 (2011) pp. 1105-1113.