2023 - ISU Putnam Practice Set 11 - Solutions

Thursday, November 16, 2023

B-65

1. Evaluate
$$\lim_{n\to\infty}\int_0^1\int_0^1\cdots\int_0^1 \cos^2\left(\frac{\pi}{2n}(x_1+x_2+\cdots+x_n)\right)dx_1dx_2\ldots dx_n.$$

   Solution: The change of variables $x_k \mapsto 1-x_k$ yields
$$\int_0^1\int_0^1\cdots\int_0^1 \cos^2\left(\frac{\pi}{2n}(x_1+x_2+\cdots+x_n)\right)dx_1dx_2\ldots dx_n$$
$$=\int_0^1\int_0^1\cdots\int_0^1 \sin^2\left(\frac{\pi}{2n}(x_1+x_2+\cdots+x_n)\right)dx_1dx_2\ldots dx_n.$$

   Each of these being half the sum, which is 1, shows that each integral is $\frac{1}{2}$ and so is the limit.

2. In a round-robin tournament with n players P_1, P_2,..., P_n (where $n>1$), each player plays one game with each of the other players and the rules are such that no ties can occur. Let w_i and ℓ_i, be the number of games won and lost, respectively, by P_i. Show
$$\sum_{i=1}^n w_i^2 = \sum_{i=1}^n \ell_i^2.$$

   Solution (Putnam 1965 B-2): Clearly $w_i+\ell_i = n-1$ for $i=1,2,\ldots,n$, and $\sum_{i=1}^n w_i = \sum_{i=1}^n \ell_i = \binom{n}{2}$. Hence
$$\sum_{i=1}^n w_r^2 - \sum_{i=1}^n \ell_i^2 = \sum_{i=1}^n (w_i+\ell_i)(w_i-\ell_i) = (n-1)\sum_{i=1}^n (w_i-\ell_i) = (n-1)\cdot 0 = 0.$$

3. Prove that there are exactly three right-angled triangles whose sides are integers while the area is numerically equal to twice the perimeter.

   Solution (Putnam 1965 B-3): All Pythagorean triples can be obtained from
$$X=\lambda(p^2-q^2), \qquad y=2\lambda pq, \qquad z=\lambda(p2+q2),$$

   where $O<q<p$, $\gcd(p,q)=1$ and $p\equiv q(\text{mod}2)$, λ being any natural number.

The problem requires that $\frac{1}{2}xy = 2(x+y+z)$. This condition can be written

$$\lambda^2(p^2-q^2)(pq) = 2\lambda(p^2-q^2+2pq+p^2+q^2)$$

or simply

$$\lambda(p-q)q = 4.$$

Since $p-q$ is odd it follows that $p-q=1$ and the only possibilities for q are 1, 2, 4. If

$$q=1, p=2, \qquad \lambda=4, x=12, y=16, z=20.$$
$$q=2, p=3, \qquad \lambda=2, x=10, y=24, z=26.$$
$$q=4, p=5, \qquad \lambda=1, x=9, y=40, z=41.$$

4. Consider collections of unordered pairs of V different objects $a,b,c,\ldots,$. Three pairs such as bc, ca, ab are said to form a triangle. Prove that, if $4E \le V^2$, it is possible to choose E pairs so that no triangle is formed.

   Solution (Putnam 1965 B-5): Divide the objects into two subsets $\{a_1,a_2,\ldots,a_m\}$ and $\{b_1,b2,\ldots,b_n\}$ where $m+n=V$. Then the mn pairs (a_j,b_k), where $j=1,2,\ldots,m$ and $k=1,2,\ldots,n$, obviously contain no triangles. If $|V|$ is even, take $m=n=V/2$, and if V is odd, take $m=(V+1)/2$, $n=(V-1)/2$. Then $mn \ge \lfloor V^2/4 \rfloor \ge E$.

5. If A, B, C, D are four distinct points such that every circle through A and B intersects (or coincides with) every circle through C and D, prove that the four points are either collinear (all of one line) or concyclic (all on one circle).

   Solution (Putnam 1965 B-6): Suppose A, B, C, D are neither concyclic nor collinear. Then p, the perpendicular bisector of segment AB, cannot coincide with q, the perpendicular bisector of segment CD. If the lines p and q intersect, their common point is the center of two concentric circles, one through A and B, the other through C and D. If instead p and q are parallel, so also are the lines AB and CD. Consider points P and Q, on p and q respectively, midway between the parallel lines AB and CD. Clearly, the circles ABP and CDQ have no common point.