2021 - ISU Putnam Practice Set 11 - Solutions

Wednesday, December 17, 2021

A1 Sauce

1. Show that every positive integer is a sum of one or more numbers of the form 2^r3^s, where r and s are nonnegative integers and no summand divides another. (For example, 23 = 9 + 8 + 6.)

   Solution: (2005 A1) We proceed by induction, with base case $1 = 2^0 3^0$. Suppose all integers less than $n-1$ can be represented. If n is even, then we can take a representation of $n/2$ and multiply each term by 2 to obtain a representation of n. If n is odd, put $m = \lfloor \log_3 n \rfloor$, so that $3^m \leq n < 3^{m+1}$. If $3^m = n$, we are done. Otherwise, choose a representation $(n-3^m)/2 = s_1 + \cdots + s_k$ in the desired form. Then

   $$n = 3^m + 2s_1 + \cdots + 2s_k,$$

   and clearly none of the $2s_i$ divide each other or 3^m. Moreover, since $2s_i \leq n - 3^m < 3^{m+1} - 3^m$, we have $s_i < 3^m$, so 3^m cannot divide $2s_i$ either. Thus n has a representation of the desired form in all cases, completing the induction.

   Remarks: This problem is originally due to Paul Erdős. Note that the representations need not be unique: for instance,

   $$11 = 2 + 9 = 3 + 8.$$

2. Find the volume of the region of points (x,y,z) such that

   $$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

   Solution: (2006 A1) We change to cylindrical coordinates, i.e., we put $r = \sqrt{x^2+y^2}$. Then the given inequality is equivalent to

   $$r^2 + z^2 + 8 \leq 6r,$$

   or

   $$(r-3)^2 + z^2 \leq 1.$$

This defines a solid of revolution (a solid torus); the area being rotated is the disc $(x-3)^2+z^2 \leq 1$ in the xz-plane. By Pappus's theorem, the volume of this equals the area of this disc, which is π, times the distance through which the center of mass is being rotated, which is $(2\pi)3$. That is, the total volume is $6\pi^2$.

3. Find all values of α for which the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$ are tangent to each other.

Solution: (2007 A1) The only such α are $2/3, 3/2, (13 \pm \sqrt{601})/12$.

First solution: Let C_1 and C_2 be the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$, respectively, and let L be the line $y = x$. We consider three cases.

If C_1 is tangent to L, then the point of tangency (x,x) satisfies

$$2\alpha x + \alpha = 1, \qquad x = \alpha x^2 + \alpha x + \frac{1}{24};$$

by symmetry, C_2 is tangent to L there, so C_1 and C_2 are tangent. Writing $\alpha = 1/(2x+1)$ in the first equation and substituting into the second, we must have

$$x = \frac{x^2+x}{2x+1} + \frac{1}{24},$$

which simplifies to $0 = 24x^2 - 2x - 1 = (6x+1)(4x-1)$, or $x \in \{1/4, -1/6\}$. This yields $\alpha = 1/(2x+1) \in \{2/3, 3/2\}$.

If C_1 does not intersect L, then C_1 and C_2 are separated by L and so cannot be tangent.

If C_1 intersects L in two distinct points P_1, P_2, then it is not tangent to L at either point. Suppose at one of these points, say P_1, the tangent to C_1 is perpendicular to L; then by symmetry, the same will be true of C_2, so C_1 and C_2 will be tangent at P_1. In this case, the point $P_1 = (x,x)$ satisfies

$$2\alpha x + \alpha = -1, \qquad x = \alpha x^2 + \alpha x + \frac{1}{24};$$

writing $\alpha = -1/(2x+1)$ in the first equation and substituting into the second, we have

$$x = -\frac{x^2+x}{2x+1} + \frac{1}{24},$$

or $x = (-23 \pm \sqrt{601})/72$. This yields $\alpha = -1/(2x+1) = (13 \pm \sqrt{601})/12$.

If instead the tangents to C_1 at P_1, P_2 are not perpendicular to L, then we claim there cannot be any point where C_1 and C_2 are tangent. Indeed, if we count intersections of C_1 and C_2 (by using C_1 to substitute for y in C_2, then solving for y), we get at most four solutions counting multiplicity. Two of these are P_1 and P_2, and any point of tangency counts for two more. However, off of L, any point of tangency would have a mirror image which is also a point of tangency, and there cannot be six solutions. Hence we have now found all possible α.

Second solution: For any nonzero value of α, the two conics will intersect in four points in the complex projective plane $\mathbb{P}^2(\mathbb{C})$. To determine the y-coordinates of these intersection points, subtract the two equations to obtain

$$(y-x) = \alpha(x-y)(x+y) + \alpha(x-y).$$

Therefore, at a point of intersection we have either $x = y$, or $x = -1/\alpha - (y+1)$. Substituting these two possible linear conditions into the second equation shows that the y-coordinate of a point of intersection is a root of either $Q_1(y) = \alpha y^2 + (\alpha - 1)y + 1/24$ or $Q_2(y) = \alpha y^2 + (\alpha+1)y + 25/24 + 1/\alpha$.

If two curves are tangent, then the y-coordinates of at least two of the intersection points will coincide; the converse is also true because one of the curves is the graph of a function in x. The coincidence occurs precisely when either the discriminant of at least one of Q_1 or Q_2 is zero, or there is a common root of Q_1 and Q_2. Computing the discriminants of Q_1 and Q_2 yields (up to constant factors) $f_1(\alpha) = 6\alpha^2 - 13\alpha + 6$ and $f_2(\alpha) = 6\alpha^2 - 13\alpha - 18$, respectively. If on the other hand Q_1 and Q_2 have a common root, it must be also a root of $Q_2(y) - Q_1(y) = 2y + 1 + 1/\alpha$, yielding $y = -(1+\alpha)/(2\alpha)$ and $0 = Q_1(y) = -f_2(\alpha)/(24\alpha)$.

Thus the values of α for which the two curves are tangent must be contained in the set of zeros of f_1 and f_2, namely $2/3$, $3/2$, and $(13 \pm \sqrt{601})/12$.

Remark: The fact that the two conics in $\mathbb{P}^2(\mathbb{C})$ meet in four points, counted with multiplicities, is a special case of *Bézout's theorem*: two curves in $\mathbb{P}^2(\mathbb{C})$ of degrees m, n and not sharing any common component meet in exactly mn points when counted with multiplicity.

Many solvers were surprised that the proposers chose the parameter $1/24$ to give two rational roots and two nonrational roots. In fact, they had no choice in the matter: attempting to make all four roots rational by replacing $1/24$ by β amounts to asking for $\beta^2 + \beta$ and

$\beta^2+\beta+1$ to be perfect squares. This cannot happen outside of trivial cases ($\beta = 0,-1$) ultimately because the elliptic curve 24A1 (in Cremona's notation) over $\mathbb{Q}$ has rank 0. (Thanks to Noam Elkies for providing this computation.)

However, there are choices that make the radical milder, e.g., $\beta = 1/3$ gives $\beta^2+\beta = 4/9$ and $\beta^2+\beta+1 = 13/9$, while $\beta = 3/5$ gives $\beta^2+\beta = 24/25$ and $\beta^2+\beta+1 = 49/25$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x,y)+f(y,z)+f(z,x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = g(x)-g(y)$ for all real numbers x and y.

   Solution: (2008 A1) The function $g(x) = f(x,0)$ works. Substituting $(x,y,z) = (0,0,0)$ into the given functional equation yields $f(0,0) = 0$, whence substituting $(x,y,z) = (x,0,0)$ yields $f(x,0)+f(0,x) = 0$. Finally, substituting $(x,y,z) = (x,y,0)$ yields $f(x,y) = -f(y,0)-f(0,x) = g(x)-g(y)$.

   Remark: A similar argument shows that the possible functions g are precisely those of the form $f(x,0)+c$ for some c.

5. Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A)+f(B)+f(C)+f(D) = 0$. Does it follow that $f(P) = 0$ for all points P in the plane?

   Solution: (2009 A1) Yes, it does follow. Let P be any point in the plane. Let $ABCD$ be any square with center P. Let E,F,G,H be the midpoints of the segments AB,BC,CD,DA, respectively. The function f must satisfy the equations

$$\begin{aligned}
0 &= f(A)+f(B)+f(C)+f(D)\\
0 &= f(E)+f(F)+f(G)+f(H)\\
0 &= f(A)+f(E)+f(P)+f(H)\\
0 &= f(B)+f(F)+f(P)+f(E)\\
0 &= f(C)+f(G)+f(P)+f(F)\\
0 &= f(D)+f(H)+f(P)+f(G).
\end{aligned}$$

   If we add the last four equations, then subtract the first equation and twice the second equation, we obtain $0 = 4f(P)$, whence $f(P) = 0$.

   Remark. Problem 1 of the 1996 Romanian IMO team selection exam asks the same question with squares replaced by regular polygons of any (fixed) number of vertices.