1. The triangle ABC has an obtuse angle at B, and angle A is less than angle C. The external angle bisector at A meets the line BC at D, and the external angle bisector at B meets the line AC at E. Also, BA = AD = BE. Find angle A.

Solution (Putnam 1965 A-1): Let angle BAC = k. Then since BA = BE, angle BEA = k. Take B’ on BA the opposite side of B to A. Then angle B’BE = 2k. Angle B’BC is bisected by BE, so angle CBE = 2k. Hence angle ACB = 3k. So angle DBA = 4k. But AD = BA, so angle BDA = 4k. But AD is the exterior bisector, so angle BAD = 90 - k/2. The angles in BAD must sum to 180 deg, so k = 12 deg.

2. Let $\alpha$ and $\beta$ be positive real numbers such that $1/\alpha + 1/\beta = 1$. Prove that the line $mx + ny = 1$ with $m, n$ positive reals is tangent to the curve $x^{\alpha} + y^{\alpha} = 1$ in the first quadrant $(x, y \geq 0)$ iff $m^\beta + n^\beta = 1$.

Solution (Putnam 1965 A-6): Suppose $mx + ny = 1$ is tangent to the curve. Suppose it touches at $(a, b)$. Differentiating, we see that the tangent at $(a, b)$ is $a^{\alpha - 1}x + b^{\alpha - 1}y = 1$, so $m = a^{\alpha - 1}, n = b^{\alpha - 1}$. Hence, using $\alpha \beta - \beta = \alpha$, we have that $m^\beta + n^\beta = a^\alpha + b^\alpha = 1$.

Conversely, suppose that $m^\beta + n^\beta = 1$. Take $a = m^\beta/\alpha, b = n^\beta/\alpha$. Then $a^\alpha + b^\alpha = 1$, so $(a, b)$ lies on the curve in the first quadrant. Its tangent is $Mx + Ny = 1$, where $M = a^{\alpha - 1}, N = b^{\beta - 1}$. But $a = m^\beta/\alpha$ and $\beta/\alpha (\alpha - 1) = 1$, so $M = m$. Similarly, $N = n$. Thus we have established that $mx + ny = 1$ is tangent to the curve in the first quadrant as required.

3. Show that, for any positive integer $n$,

$$\sum_{r=0}^{\lfloor(n-1)/2\rfloor} \binom{n-2r}{n}^2 = \frac{1}{n} \binom{2n-2}{n-1},$$

where $\lfloor x \rfloor$ means the greatest integer not exceeding $x$, and $\binom{n}{r}$ is the binomial coefficient ”$n$ choose $r$,” with the convention $\binom{n}{0} = 1$. 
Solution (Putnam 1965 A-2): Substituting $s = n - r$ in the given summation reveals that twice this sum is equal to:

$$\sum_{r=0}^{n} \left( \frac{n-2r}{n} \binom{n}{r} \right)^2 = \sum_{r=0}^{n} \left( 1 - 2 \frac{r}{n} \right)^2 \binom{n}{r}^2$$

$$= \sum_{r=0}^{n} \binom{n}{r}^2 - 4 \sum_{r=0}^{n} \frac{r}{n} \binom{n}{r} \binom{n}{r} + 4 \sum_{r=0}^{n} \frac{r}{n}^2 \binom{n}{r}^2$$

$$= \binom{2n}{n} - 4 \sum_{r=0}^{n} \binom{n-1}{r-1} \binom{n}{r} + 4 \sum_{r=0}^{n} \binom{n-1}{r-1}^2$$

$$= \binom{2n}{n} - 4 \binom{2n-1}{n-1} + 4 \binom{2n-2}{n-1}$$

$$= \binom{2n}{n} - 4 \binom{2n-2}{n-2}$$

$$= \binom{2n(2n-1)}{n^2} - 4 \frac{n-1}{n} \binom{2n-2}{n-1}$$

$$= 2 \binom{2n-2}{n-1}.$$

This assume the well-known identites

$$\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n} \quad \text{and} \quad \sum_{r=0}^{k} \binom{m}{k-r} \binom{n}{r} = \binom{m+n}{k}$$

which may be proved by comparing the coefficients in the expansion of

$$(1+x)^m(1+x)^n = (1+x)^{m+n}.$$ 

4. $S$ and $T$ are finite sets. $U$ is a collection of ordered pairs $(s, t)$ with $s \in S$ and $t \in T$. There is no element $s \in S$ such that all possible pairs $(s, t) \in U$. Every element $t \in T$ appears in at least one element of $U$. Prove that we can find distinct $s_1, s_2 \in S$ and distinct $t_1, t_2 \in T$ such that $(s_1, t_1), (s_2, t_2) \in U$, but $(s_1, t_2), (s_2, t_1) \notin U$.

Solution (Putnam 1965 A-4): Suppose that we cannot find such $s_i, t_i$. We will establish a contradiction.

Take $t$ in $T$. Suppose that there are $n$ distinct $s$ in $S$ such that $(s, t)$ is in $U$. Suppose $n > 0$. Then take a specific $s'$ such that $(s', t)$ is in $U$. There must be some $t'$ such that $(s', t')$ is not in $U$. Now consider whether we have $(s, t')$ in $U$. If $(s, t)$ is not in $U$, then $(s, t')$ cannot be in $U$ (or we would have found $s_i, t_i$). But there are at most $n - 1$ distinct $s$ such that $(s, t')$ is
in $U$ (the only candidates are the cases for which $(s,t)$ is in $U$, and one of those, namely $s'$, does not work).

Iterating, we must eventually get some $x$ in $T$ for which there is no $s$ in $S$ with $(s,x)$ in $U$. Contradiction.

5. How many possible bijections $f$ on $\{1,2,\ldots,n\}$ are there such that for each $i = 2, 3, \ldots, n$ we can find $j < n$ with $f(i) - f(j) = \pm 1$?

**Solution (Putnam 1965 A-5):** $2^{n-1}$

Consider the last element $f(n)$. Suppose it is $m$, not 1 or $n$. Then the earlier elements fall into two non-empty sets $A = \{1,2,\ldots,m-1\}$ and $B = \{m+1,m+2,\ldots,n\}$. But the difference between an element of $A$ and an element of $B$ is at least 2. So if $f(1)$ is in $A$, then the first time we get an element of $B$ it has only elements of $A$ preceding it. Contradiction. Similarly, if $f(1)$ is in $B$.

So we conclude that the last element is always 1 or $n$. We can now prove the result by induction. Clearly given an arrangement for $n$ we can derive one for $n+1$ by adding $n+1$ at the end. We can also derive one for $n+1$ by increasing each element by 1 and adding 1 at the end. Equally it is clear that all these are distinct and that there are no other arrangements for $n+1$ that end in 1 or $n+1$. So there are twice as many arrangements for $n+1$ as for $n$. 