2023 - ISU Putnam Practice Set 10 - Solutions

Thursday, November 9, 2023

26

1. The triangle ABC has an obtuse angle at B, and angle A is less than angle C. The external angle bisector at A meets the line BC at D, and the external angle bisector at B meets the line AC at E. Also, BA = AD = BE. Find angle A.

   Solution (Putnam 1965 A-1): Let angle BAC = k. Then since BA = BE, angle BEA = k. Take B' on BA the opposite side of B to A. Then angle B'BE = 2k. Angle B'BC is bisected by BE, so angle CBE = 2k. Hence angle ACB = 3k. So angle DBA = 4k. But AD = BA, so angle BDA = 4k. But AD is the exterior bisector, so angle BAD = 90 - k/2. The angles in BAD must sum to 180 deg, so k = 12 deg.

2. Let α and β be positive real numbers such that $1/\alpha + 1/\beta = 1$. Prove that the line $mx + ny = 1$ with m, n positive reals is tangent to the curve $x^\alpha + y^\alpha = 1$ in the first quadrant $(x, y \geq 0)$ iff $m^\beta + n^\beta = 1$.

   Solution (Putnam 1965 A-6): Suppose $mx + ny = 1$ is tangent to the curve. Suppose it touches at (a, b). Differentiating, we see that the tangent at (a, b) is $a^{\alpha-1}x + b^{\alpha-1}y = 1$, so $m = a^{\alpha-1}$, $n = b^{\alpha-1}$. Hence, using $\alpha\beta - \beta = \alpha$, we have that $m^\beta + n^\beta = a^\alpha + b^\alpha = 1$.

   Conversely, suppose that $m^\beta + n^\beta = 1$. Take $a = m^{\beta/\alpha}$, $b = n^{\beta/\alpha}$. Then $a^\alpha + b^\alpha = 1$, so (a, b) lies on the curve in the first quadrant. Its tangent is $Mx + Ny = 1$, where $M = a^{\alpha-1}$, $N = b^{\beta-1}$. But $a = m^{\beta/\alpha}$ and $\beta/\alpha(\alpha - 1) = 1$, so $M = m$. Similarly, $N = n$. Thus we have established that $mx + ny = 1$ is tangent to the curve in the first quadrant as required.

3. Show that, for any positive integer n,

$$\sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \left(\frac{n-2r}{n} \binom{n}{r} \right)^2 = \frac{1}{n} \binom{2n-2}{n-1},$$

   where $\lfloor x \rfloor$ means the greatest integer not exceeding x, and $\binom{n}{r}$ is the binomial coefficient "n choose r," with the convention $\binom{n}{0} = 1$.

Solution (Putnam 1965 A-2): Substituting $s = n - r$ in the given summation reveals that twice this sum is equal to:

$$\begin{aligned}
\sum_{r=0}^{n}\left(\frac{n-2r}{n}\binom{n}{r}\right)^2 &= \sum_{r=0}^{n}\left(1-2\frac{r}{n}\right)^2\binom{n}{r}^2 \\
&= \sum_{r=0}^{n}\binom{n}{r}^2 - 4\sum_{r=0}^{n}\frac{r}{n}\binom{n}{r}\binom{n}{r} + 4\sum_{r=0}^{n}\left(\frac{r}{n}\right)^2\binom{n}{r}^2 \\
&= \binom{2n}{n} - 4\sum_{r=0}^{n}\binom{n-1}{r-1}\binom{n}{r} + 4\sum_{r=0}^{n}\binom{n-1}{r-1}^2 \\
&= \binom{2n}{n} - 4\binom{2n-1}{n-1} + 4\binom{2n-2}{n-1} \\
&= \binom{2n}{n} - 4\binom{2n-2}{n-2} \\
&= \left(\frac{2n(2n-1)}{n^2} - 4\frac{n-1}{n}\right)\binom{2n-2}{n-1} \\
&= \frac{2}{n}\binom{2n-2}{n-1}.
\end{aligned}$$

This assume the well-known identites

$$\sum_{r=0}^{n}\binom{n}{r}^2 = \binom{2n}{n} \qquad \text{and} \qquad \sum_{r=0}^{k}\binom{m}{k-r}\binom{n}{r} = \binom{m+n}{k}$$

which may be proved by comparing the coefficients in the expansion of

$$(1+x)^m(1+x)^n = (1+x)^{m+n}.$$

4. S and T and finite sets. U is a collection of ordered pairs (s,t) with $s \in S$ and $t \in T$. There is no element $s \in S$ such that all possible pairs $(s,t) \in U$. Every element $t \in T$ appears in at least one element of U. Prove that we can find distinct $s_1, s_2 \in S$ and distinct $t_1, t_2 \in T$ such that $(s_1,t_1),(s_2,t_2) \in U$, but $(s_1,t_2),(s_2,t_1) \notin U$.

   Solution (Putnam 1965 A-4): Suppose that we cannot find such s_i, t_i. We will establish a contradiction.

   Take t in T. Suppose that there are n distinct s in S such that (s,t) is in U. Suppose $n > 0$. Then take a specific s' such that (s',t) is in U. There must be some t' such that (s',t') is not in U. Now consider whether we have (s,t') in U. If (s,t) is not in U, then (s,t') cannot be in U (or we would have found s_i, t_i). But there are at most $n-1$ distinct s such that (s,t') is in U (the only candidates are the cases for which (s,t) is in U, and one of those, namely s', does not work).

Iterating, we must eventually get some x in T for which there is no s in S with (s,x) in U. Contradiction.

5. How many possible bijections f on $\{1,2,\ldots,n\}$ are there such that for each $i = 2,3,\ldots,n$ we can find $j < n$ with $f(i) - f(j) = \pm 1$?

   Solution (Putnam 1965 A-5): 2^{n-1}

   Consider the last element $f(n)$. Suppose it is m, not 1 or n. Then the earlier elements fall into two non-empty sets $A = \{1,2,\ldots,m-1\}$ and $B = \{m+1,m+2,\ldots,n\}$. But the difference between an element of A and an element of B is at least 2. So if $f(1)$ is in A, then the first time we get an element of B it has only elements of A preceding it. Contradiction. Similarly, if $f(1)$ is in B.

   So we conclude that the last element is always 1 or n. We can now prove the result by induction. Clearly given an arrangement for n we can derive one for $n+1$ by adding $n+1$ at the end. We can also derive one for $n+1$ by increasing each element by 1 and adding 1 at the end. Equally it is clear that all these are distinct and that there are no other arrangements for $n+1$ that end in 1 or $n+1$. So thee are twice as many arrangements for $n+1$ as for n.