2021 - ISU Putnam Practice Set 10 - Solutions

Wednesday, December 2, 2022

B1 Bingo!

1. For a partition π of $\{1,2,3,4,5,6,7,8,9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1,2,3,4,5,6,7,8,9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

   Solution Putnam 1995 B1: For a given π, no more than three different values of $\pi(x)$ are possible (four would require one part each of size at least 1,2,3,4, and that's already more than 9 elements). If no such x, y exist, each pair $(\pi(x), \pi'(x))$ occurs for at most 1 element of x, and since there are only 3×3 possible pairs, each must occur exactly once. In particular, each value of $\pi(x)$ must occur 3 times. However, clearly any given value of $\pi(x)$ occurs $k\pi(x)$ times, where k is the number of distinct partitions of that size. Thus $\pi(x)$ can occur 3 times only if it equals 1 or 3, but we have three distinct values for which it occurs, contradiction.

2. Find the minimum value of

$$\frac{(x+1/x)^6 - (x^6 + 1/x^6) - 2}{(x+1/x)^3 + (x^3 + 1/x^3)}$$

   for $x > 0$.

   Solution Putnam 1998 B1: Notice that

$$\frac{(x+1/x)^6 - (x^6 + 1/x^6) - 2}{(x+1/x)^3 + (x^3 + 1/x^3)} =$$
$$(x+1/x)^3 - (x^3 + 1/x^3) = 3(x+1/x)$$

   (difference of squares). The latter is easily seen (e.g., by AM-GM) to have minimum value 6 (achieved at $x = 1$).

3. Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta \to 0} |EF|$.

Solution Putnam 1999 B1: The answer is 1/3. Let G be the point obtained by reflecting C about the line AB. Since $\angle ADC = \frac{\pi-\theta}{2}$, we find that $\angle BDE = \pi - \theta - \angle ADC = \frac{\pi-\theta}{2} = \angle ADC = \pi - \angle BDC = \pi - \angle BDG$, so that E, D, G are collinear. Hence

$$|EF| = \frac{|BE|}{|BC|} = \frac{|BE|}{|BG|} = \frac{\sin(\theta/2)}{\sin(3\theta/2)},$$

where we have used the law of sines in $\triangle BDG$. But by l'Hôpital's Rule,

$$\lim_{\theta\to 0} \frac{\sin(\theta/2)}{\sin(3\theta/2)} = \lim_{\theta\to 0} \frac{\cos(\theta/2)}{3\cos(3\theta/2)} = 1/3.$$

4. Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

Solution Putnam 2003 B1: Suppose the contrary. By setting $y = -1, 0, 1$ in succession, we see that the polynomials $1 - x + x^2, 1, 1 + x + x^2$ are linear combinations of $a(x)$ and $b(x)$. But these three polynomials are linearly independent, so cannot all be written as linear combinations of two other polynomials, contradiction.

5. What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

Solution Putnam 2008 B1: There are at most two such points. For example, the points $(0,0)$ and $(1,0)$ lie on a circle with center $(1/2, x)$ for any real number x, not necessarily rational.

On the other hand, suppose $P = (a,b), Q = (c,d), R = (e,f)$ are three rational points that lie on a circle. The midpoint M of the side PQ is $((a+c)/2, (b+d)/2)$, which is again rational. Moreover, the slope of the line PQ is $(d-b)/(c-a)$, so the slope of the line through M perpendicular to PQ is $(a-c)/(b-d)$, which is rational or infinite.

Similarly, if N is the midpoint of QR, then N is a rational point and the line through N perpendicular to QR has rational slope. The center of the circle lies on both of these lines, so its coordinates (g,h) satisfy two linear equations with rational coefficients, say $Ag + Bh = C$

and $Dg + Eh = F$. Moreover, these equations have a unique solution. That solution must then be

$$g = (CE - BD)/(AE - BD)$$
$$h = (AF - BC)/(AE - BD)$$

(by elementary algebra, or Cramer's rule), so the center of the circle is rational. This proves the desired result.