2021 - ISU Putnam Practice Set 10 - Solutions

Wednesday, December 1, 2021

Graphs

1. Three conflicting neighbors have three common wells. Can one draw nine paths con- necting each of the neighbors to each of the wells such that no two paths intersect?

   Solution: No. If this were possible, then the configuration would determine a planar graph with $V = 6$ vertices (the 3 neighbors and the 3 wells) and $E = 9$ edges (the paths). Each of its F faces would have 4 or more edges because there is no path between wells or between neighbors. So

   $$F \leq \frac{2}{4}E = \frac{9}{2}.$$

   On the other hand, by Euler's relations we have

   $$F = 2 + E - V = 5,$$

   a contradiction.

2. In a tournament 2n teams took part. On the first day, n pairs of teams competed. On the second day, other n pairs of teams competed. Show that at the end of the second day one can find n teams such that no two have competed with each other yet.

   Solution: Let the 2n teams be the vertices of a graph. Draw a red edge for pairs that competed on the first day, and a blue edge for the pairs that competed on the second day. Then each vertex belongs to exactly a red edge and exactly a blue edge. We now have a graph consisting of cycles, each of which having an even number of vertices. Choose every other vertex from each cycle to obtain the desired set of n teams that have not played with each other.

3. Let G be a connected graph with k edges. Show that it is possible to label the edges of this graph with the numbers 1, 2, . . . , k, so that for every vertex that belongs to at least two edges, the greatest common divisor of the integers that label the edges containing this vertex is equal to 1.

Solution: Recall that the degree of a vertex is the number of edges containing it. If G has some vertices of odd degree, the number of such vertices is even because the sum of the degrees of all vertices equals twice the number of edges. In this situation we add a vertex to G and connect it by edges to all vertices of odd degree. The new graph, G' has all vertices of even degree, therefore G' has an Eulerian cycle. We label the edges of G by 1, 2, . . . , k in the order in which we encounter these when traveling on the Eulerian cycle. When passing through a vertex of G of even degree, two edges are labeled by consecutive numbers, hence this vertex will have the desired property. On the other hand, we are only interested in vertices of G of odd degree that have a degree greater than or equal to 3. Through one such vertex we pass at least twice, and only once do we pass through it on edges that don?t belong to G (since there is only one such edge). Hence again there are two edges labeled by consecutive integers. The problem is solved.

4. Consider a convex polyhedron whose faces are triangles and whose edges are oriented. A singularity is a face whose edges form a cycle, a vertex that belongs only to incoming edges, or a vertex that belongs only to outgoing edges. Show that the polyhedron has at least two singularities.

   Solution: We will prove a more precise result. To this end, we need to define one more type of singularity. A vertex is called a (multi)saddle of index $-k$, $k \geq 1$, if it belongs to some incoming and to some outgoing edge, and if there are k + 1 changes from incoming to outgoing edges in making a complete turn around the vertex. The name is motivated by the fact that if the index is -1, then the arrows describe the way liquid flows on a horse saddle.

   Call a vertex that belongs only to outgoing edges a source, a vertex that belongs only to incoming edges a sink, and a face whose edges form a cycle a circulation. Denote by n_1 the number of sources, by n_2 the number of sinks, by n_3 the number of circulations, and by n_4 the sum of the indices of all (multi)saddles.

   We start with the count of vertices by incoming edges; thus for each incoming edge we count one vertex. Sources are not counted. With the standard notation, if we write

$$E = V - n_1,$$

   we have overcounted on the left-hand side. To compensate this, let us count vertices by faces. Each face that is not a circulation has two edges pointing toward the same vertex. In that case, for that face we count that vertex. All faces but the circulations count, and for vertices that are not singularities this takes care of the overcount. So we can improve our "equality' to

$$E = V - n_1 + F - n_3.$$

Each sink is overcounted by 1 on the right. We improve again to

$$E = V - n_1 + F - n_3 - n_2.$$

Still, the right-hand side undercounts saddles, and each saddle is undercounted by the absolute value of its index. We finally reach equality with

$$E = V - n_1 + F - n_3 - n_2 + |n_4| = V + F - n_1 - n_2 - n_3 - n_4.$$

Using Eulerss formula, we obtain

$$n_1 + n_2 + n_3 + n_4 = V - E + F = 2.$$

Because $n_4 \leq 0$, we have $n_1 + n_2 + n_3 \geq 2$, which is what we had to prove.

5. Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

> Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

Show that the player who signs first will always win by playing as well as possible.

Solution: (2002 B2) (Note: the problem statement assumes that all polyhedra are connected and that no two edges share more than one face, so we will do likewise. In particular, these are true for all convex polyhedra.) We show that in fact the first player can win on the third move. Suppose the polyhedron has a face A with at least four edges. If the first player plays there first, after the second player's first move there will be three consecutive faces B, C, D adjacent to A which are all unoccupied. The first player wins by playing in C; after the second player's second move, at least one of B and D remains unoccupied, and either is a winning move for the first player.

It remains to show that the polyhedron has a face with at least four edges. (Thanks to Russ Mann for suggesting the following argument.) Suppose on the contrary that each face has only three edges. Starting with any face F_1 with vertices v_1, v_2, v_3, let v_4 be the other endpoint of the third edge out of v_1. Then the faces adjacent to F_1 must have vertices v_1, v_2, v_4; v_1, v_3, v_4; and v_2, v_3, v_4. Thus v_1, v_2, v_3, v_4 form a polyhedron by themselves, contradicting the fact that the given polyhedron is connected and has at least five vertices. (One can also deduce this using Euler's formula $V - E + F = 2 - 2g$, where V, E, F are the numbers of vertices, edges and faces, respectively, and g is the genus of the polyhedron. For a convex polyhedron, $g = 0$ and you get the "usual" Euler's formula.)

Note: Walter Stromquist points out the following counterexample if one relaxes the assumption that a pair of faces may not share multiple edges. Take a tetrahedron and remove a smaller tetrahedron from the center of an edge; this creates two small triangular faces and turns two of the original faces into hexagons. Then the second player can draw by signing one of the hexagons, one of the large triangles, and one of the small triangles. (He does this by "mirroring": wherever the first player signs, the second player signs the other face of the same type.)