2023 - ISU Putnam Practice Set 9 - Solutions

Thursday, November 2, 2023

Series

1. Suppose that a sequence $a_1, a_2, a_3, \dots$ satisfies $0 < a_n \le a_{2n} + a_{2n+1}$ for all $n \ge 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

   Solution (Putnam 1994 A-1): If it converges then

   $$\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=1}^{\infty}(a_{2n} + a_{2n+1}) > a_1 + \sum_{n=1}^{\infty} a_n,$$

   which is impossible.

2. For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

   $$\sum_{n=1}^{2013} c(n)c(n+2).$$

   Solution (Putnam 2013 B-1): Note that

   $$\begin{aligned} c(2k+1)c(2k+3) &= (-1)^k c(k)(-1)^{k+1} c(k+1) \\ &= -c(k)c(k+1) \\ &= -c(2k)c(2k+2). \end{aligned}$$

   It follows that $\sum_{n=2}^{2013} c(n)c(n+2) = \sum_{k=1}^{1006}(c(2k)c(2k+2) + c(2k+1)c(2k+3)) = 0$, and so the desired sum is $c(1)c(3) = -1$.

3. Evaluate

   $$\int_0^{\infty} \left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots\right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right) dx.$$

   Solution (Putnam 1997 A-3): Note that the series on the left is simply $x\exp(-x^2/2)$. By integration by parts,

   $$\int_0^{\infty} x^{2n+1} e^{-x^2/2} dx = 2n \int_0^{\infty} x^{2n-1} e^{-x^2/2} dx$$

and so by induction,

$$\int_0^\infty x^{2n+1} e^{-x^2/2} dx = 2 \times 4 \times \cdots \times 2n.$$

Thus the desired integral is simply

$$\sum_{n=0}^{\infty} \frac{1}{2^n n!} = \sqrt{e}.$$

4. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \leq e \\ x f(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

Solution (Putnam 2008 A-4): The sum diverges. From the definition, $f(x) = x$ on $[1, e]$, $x \ln x$ on $(e, e^e]$, $x \ln x \ln \ln x$ on $(e^e, e^{e^e}]$, and so forth. It follows that on $[1, \infty)$, f is positive, continuous, and increasing. Thus $\sum_{n=1}^{\infty} \frac{1}{f(n)}$, if it converges, is bounded below by $\int_1^\infty \frac{dx}{f(x)}$; it suffices to prove that the integral diverges.

Write $\ln^1 x = \ln x$ and $\ln^k x = \ln(\ln^{k-1} x)$ for $k \geq 2$; similarly write $\exp^1 x = e^x$ and $\exp^k x = e^{\exp^{k-1} x}$. If we write $y = \ln^k x$, then $x = \exp^k y$ and $dx = (\exp^k y)(\exp^{k-1} y) \cdots (\exp^1 y) dy = x(\ln^1 x) \cdots (\ln^{k-1} x) dy$. Now on $[\exp^{k-1} 1, \exp^k 1]$, we have $f(x) = x(\ln^1 x) \cdots (\ln^{k-1} x)$, and thus substituting $y = \ln^k x$ yields

$$\int_{\exp^{k-1} 1}^{\exp^k 1} \frac{dx}{f(x)} = \int_0^1 dy = 1.$$

It follows that $\int_1^\infty \frac{dx}{f(x)} = \sum_{k=1}^{\infty} \int_{\exp^{k-1} 1}^{\exp^k 1} \frac{dx}{f(x)}$ diverges, as desired.

5. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1} a_n - 2$ for $n = 2, 3, \ldots$. Assume that the sequence (b_j) is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \ldots a_n}$$

converges, and evaluate S.

Solution (Putnam 2011 A-2): For $m \geq 1$, write

$$S_m = \frac{3}{2} \left(1 - \frac{b_1 \cdots b_m}{(b_1 + 2) \cdots (b_m + 2)} \right).$$

Then $S_1 = 1 = 1/a_1$ and a quick calculation yields

$$S_m - S_{m-1} = \frac{b_1 \cdots b_{m-1}}{(b_2+2)\cdots(b_m+2)} = \frac{1}{a_1 \cdots a_m}$$

for $m \geq 2$, since $a_j = (b_j+2)/b_{j-1}$ for $j \geq 2$. It follows that $S_m = \sum_{n=1}^{m} 1/(a_1 \cdots a_n)$.

Now if (b_j) is bounded above by B, then $\frac{b_j}{b_j+2} \leq \frac{B}{B+2}$ for all j, and so $3/2 > S_m \geq 3/2(1 - (\frac{B}{B+2})^m)$. Since $\frac{B}{B+2} < 1$, it follows that the sequence (S_m) converges to $S = 3/2$.