2021 - ISU Putnam Practice Set 9 - Solutions

Wednesday, November 18, 2022

Probability

1. What is the probability that 3 points selected at random on a circle lie on some semicircle?

   Solution: We assume that the circle of the problem is the unit circle centered at the origin O. The space of all possible choices of three points P_1, P_2, P_3 is the product of three circles; the volume of this space is $8\pi^3$. Let us first measure the volume of the configurations (P_1, P_2, P_3) such that the arc $P_1P_2P_3$ is included in a semicircle and is oriented counterclockwise from P_1 to P_3. The condition that the arc is contained in a semicircle translates to $0 \leq angleP_1OP_2 \leq \pi$ and $0 \leq angleP_2OP_3 \leq \pi - angleP_1OP_2$. The point P_1 is chosen randomly on the circle, and for each P_1 the region of the angles θ_1 and θ_2 such that $0 \leq \theta_1 \pi$ and $0 \leq \theta_2 \leq \pi - \theta_1$ is an isosceles right triangle with leg equal to π. Hence the region of points (P_1, P_2, P_3) subject to the above constraints has volume $2\pi \cdot \frac{1}{2}\pi^2 = \pi^3$. There are $3! = 6$ such regions and they are disjoint. Therefore, the volume of the favorable region is $6\pi^3$. The desired probability is therefore $\frac{6\pi^3}{8\pi^3} = \frac{3}{4}$.

2. Let $n \geq 4$ be given, and suppose that the points $P_1, P_2, \ldots, P_n$ are randomly chosen on a circle. Consider the convex n-gon whose vertices are these points. What is the probability that at least one of the vertex angles of this polygon is acute?

   Solution: Consider the dual cube to the octahedron. The vertices A, B, C, D, E, F, G, H of this cube are the centers of the faces of the octahedron (here ABCD is a face of the cube and (A, G), (B, H), (C, E), (D, F) are pairs of diagonally opposite vertices). Each assignment of the numbers 1, 2, 3, 4, 5, 6, 7, and 8 to the faces of the octahedron corresponds to a permutation of ABCDEFGH, and thus to an octagonal circuit of these vertices. The cube has 16 diagonal segments that join nonadjacent vertices. The problem requires us to count octagonal circuits that can be formed by eight of these diagonals.

   Six of these diagonals are edges of the tetrahedron ACFH, six are edges of the tetrahedron DBEG, and four are long diagonals, joining opposite vertices of the cube. Notice that each vertex belongs to exactly one long diagonal. It follows that an octagonal circuit must contain either 2 long diagonals separated by 3 tetrahedron edges, or 4 long diagonals alternating with tetrahedron edges.

When forming a (skew) octagon with 4 long diagonals, the four tetrahedron edges need to be disjoint; hence two are opposite edges of ACFH and two are opposite edges of DBEG. For each of the three ways to choose a pair of opposite edges from the tetrahedron ACFH, there are two possible ways to choose a pair of opposite edges from tetrahedron DBEG. There are $3 \cdot 2 = 6$ octagons of this type, and for each of them, a circuit can start at 8 possible vertices and can be traced in two different ways, making a total of $6 \cdot 8 \cdot 2 = 96$ permutations.

An octagon that contains exactly two long diagonals must also contain a three-edge path along the tetrahedron ACFH and a three-edge path along tetrahedron the DBEG. A three-edge path along the tetrahedron the ACFH can be chosen in $4! = 24$ ways. The corresponding three-edge path along the tetrahedron DBEG has predetermined initial and terminal vertices; it thus can be chosen in only 2 ways. Since this counting method treats each path as different from its reverse, there are $8 \cdot 24 \cdot 2 = 384$ permutations of this type.

In all, there are $96 + 384 = 480$ permutations that correspond to octagonal circuits formed exclusively from cube diagonals. The probability of randomly choosing such a permutation is $\frac{480}{8!} = \frac{1}{84}$.

3. The temperatures in Chicago and Detroit are x° and y°, respectively. These temperatures are not assued to be independent; namely, we are given the following:

   (a) $P(x^\circ = 70^\circ) = a$

   (b) $P(y^\circ = 70^\circ) = b$

   (c) $P(\max\{x^\circ, y^\circ\} = 70^\circ) = c$.

   Determine $P(\min\{x^\circ, y^\circ\} = 70^\circ)$ in terms of a, b, c.

   Solution Putnam 1968 B1: Let A, B, C, D denote the events $x = 70^\circ$, $y = 70^\circ$, $\max(x^\circ, y^\circ) = 70^\circ$ and $\min(x^\circ, y^\circ) = 70^\circ$. Then $A \cup B = C \cup D$ and $A \cap B = C \cap D$. Hence

$$P(A) + P(B) = P(A \cup B) + P(A \cap B) = P(C \cup D) + P(C \cap D) = P(C) + P(D).$$

   Therefore $P(D) = P(A) + P(B) - P(C) = a + b - c$.

4. What is the probability that a permutation of the first n positive integers has the numbers 1 and 2 within the same cycle?

   Solution: The total number of permutations is of course $n!$. We will count instead the number of permutations for which 1 and 2 lie in different cycles. If the cycle that contains 1 has length k, we can choose the other $k-1$ elements in $\binom{n-2}{k-1}$ ways from the set $\{3,4,\ldots,n\}$. There exist $(k-1)!$ circular permutations of these elements, and $(n-k)!$ permutations of the remaining $n-k$ elements. Hence the total number of permutations for which 1 and 2 belong to different cycles is equal to

$$\sum_{k=1}^{n-1}\binom{n-2}{k-1}(k-1)!(n-k)! = (n-2)!\sum_{k=1}^{n-1}(n-k) = (n-2)!\frac{n(n-1)}{2} = \frac{n!}{2}.$$

   It follows that exactly half of all permutations contain 1 and 2 in different cycles, and so half contain 1 and 2 in the same cycle. The probability is $\frac{1}{2}$.

5. An unbiased coin is tossed n times. Find a formula, in closed form, for the expected value of $|H-T|$, where H is the number of heads, and T is the number of tails.

   Solution Putnam 1974 A4: There are $\binom{n}{k}$ ways in which exactly k tails appear, and in this case the difference is $n-2k$ Hence the expected value of $|H-T|$ is

$$\frac{1}{2^n}\sum_{k=0}^{n}|n-2k|\binom{n}{k}.$$

   Evaluate the sum as follows:

$$\begin{aligned}
\frac{1}{2^n}\sum_{k=0}^{n}|n-2k|\binom{n}{k} &= \frac{1}{2^{n-1}}\sum_{m=0}^{\lfloor n/2\rfloor}(n-2m)\binom{n}{m}\\
&= \frac{1}{2^{n-1}}\left(\sum_{m=0}^{\lfloor n/2\rfloor}(n-m)\binom{n}{m} - \sum_{m=0}^{\lfloor n/2\rfloor}m\binom{n}{m}\right)\\
&= \frac{1}{2^{n-1}}\left(\sum_{m=0}^{\lfloor n/2\rfloor}n\binom{n-1}{m} - \sum_{m=0}^{\lfloor n/2\rfloor}n\binom{n-1}{m-1}\right)\\
&= \frac{n}{2^{n-1}}\binom{n-1}{\lfloor \frac{n}{2}\rfloor}.
\end{aligned}$$