# 2021 - ISU Putnam Practice Set 9 - Solutions

## Wednesday, November 24, 2021

## Generating Functions

1. Find the general-term formula for the sequence $(y_n)_{n\geq 0}$ with $y_0 = 1$ and $y_n = ay_{n-1} + b^n$ for $n \geq 1$, where $a, b$ are fixed distinct real numbers.

   **Solution:** Let $G(x) = \sum_n y_n x^n$ be the generating function of the sequence. It satisfies the functional equation

   $$(1-ax)G(x) = 1 + bx + bx^2 + \cdots = \frac{1}{1-bx}.$$

   We find that

   $$G(x) = \frac{1}{(1-ax)(1-bx)} = \frac{A}{1-ax} + \frac{B}{1-bx} = \sum_n (Aa^n + Bb^n)x^n,$$

   for some $A$ and $B$. It follows that $y_n = Aa^n + Bb^n$, and because $y_0 = 1$ and $y_1 = a+b$, $A = \frac{a}{a-b}$ and $B = \frac{-b}{a-b}$. The general term of the sequence is therefore

   $$\frac{1}{a-b}(a^{n+1} - b^{n+1}).$$

2. Compute the sums

   $$\sum_{k=0}^{n} k\binom{n}{k} \text{ and } \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}.$$

   **Solution:** The first identity is obtained by differentiating

   $$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

   and then setting $x = 1$. The answer is $n2^{n-1}$. The second identity is obtained by integrating the same equality and then setting $x = 1$, in which case the answer is $\frac{2^{n+1}}{n+1}$.

3. Denote by $P(n)$ the number of partitions of the positive integer $n$; i.e. the number of ways of writing $n$ as a sum of positive integers. Prove that the generating function of $P(n)$, $n \geq 1$ is

   $$\sum_{n=0}^{\infty} P(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}.$$

**Solution:** We introduce some additional parameters and consider the expansion

$$\frac{1}{(1-a_1x)(1-a_2x^2)(1-a_3x^3)\cdots} = (1+a_1x+a_1^2x^2+\cdots)(1+a_2x^2+a_2^2x^4+\cdots)\cdots$$
$$= 1+a_1x+(a_1^2+a_2)x^2+\cdots+(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_k^{\lambda_k}+\cdots)x^n+\cdots$$

The term $a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_k^{\lambda_k}$ that is part of the coefficient of $x^n$ has the property that $\lambda_1+2\lambda_2+\cdots+k\lambda_k=n$; hence it defines a partition of $n$, namely,

$$n = 1+1+\cdots+1+2+\cdots+2+\cdots+k+\cdots+k,$$

where there are $\lambda_1$ ones, $\lambda_2$ twos, etc. So the terms that appear in the coefficient of $x^n$ generate all the partitions of $n$. Setting $a_1=a_2=\cdots=1$, we obtain for the coefficient of $x^n$ the number $P(n)$ of the partitions of $n$ and we are done.

4. Prove that the number of ways of writing $n$ has a sum of distinct positive integers is equal to the number of ways of writing $n$ as a sum of odd positive numbers.

   **Solution:** The argument of the previous problem can be applied mutatis mutandis to show that the number of ways of writing n as a sum of odd positive integers is the coefficient of $x_n$ in the expansion of
   $$\frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)\cdots},$$
   while the number of ways of writing $n$ as a sum of distinct positive integers is the coefficient of $x^n$ in
   $$(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$$
   We have
   $$\frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)\cdots} = \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\cdots = (1+x)(1+x^2)(1+x^3)\cdots$$
   which proves the desired equality.

5. Let $p$ be an odd prime number. Find the number of subsets of $\{1,2,\dots,p\}$ with the sum of the elements divisible by $p$

   **Solution:** The number of subsets with the sum of the elements equal to n is the coefficient of $x^n$ in the product
   $$G(x) = (1+x)(1+x^2)\cdots(1+x^p).$$

We are asked to compute the sum of the coefficients of $x^n$ for $n$ divisible by $p$. Call this number $s(p)$. There is no nice way of expanding the generating function; instead we compute $s(p)$ using particular values of $G$. It is natural to try $p$th roots of unity.

The first observation is that if $\xi$ is a $p$th root of unity, then $\sum_{k=1}^{p} \xi^p$ is zero except when $\xi = 1$. Thus if we sum the values of $G$ at the $p$th roots of unity, only those terms with exponent divisible by $p$ will survive. To be precise, if $\xi$ is a pth root of unity different from 1, then

$$\sum_{k=1}^{p} G(\xi^k) = ps(p).$$

We are left with the problem of compute $G(\xi^k)$, for $k = 1, 2, \ldots, p$. For $k = p$, this is just $2^p$. For $k = 1, 2, \ldots, p-1$,

$$G(\xi^k) = \prod_{j=1}^{p}(1+\xi^{k_j}) = \prod_{j=1}^{p}(1+\xi^j) = (-1)^p \prod_{j=1}^{p}((-1)-\xi^j) = (-1)^p((-1)^p - 1) = 2.$$

We therefore have $ps(p) = 2^p + 2(p-1) = 2^p + 2p - 2$. The answer to the problem is $s(p) = \frac{2^p-2}{p} + 2$. The expression is an integer because of Fermat's little theorem.