2023 - ISU Putnam Practice Set 8 - Solutions

Thursday, October 26, 2023

Geometry

1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

   Solution (Putnam 1998 A-1): Consider the plane containing both the axis of the cone and two opposite vertices of the cube's bottom face. The cross section of the cone and the cube in this plane consists of a rectangle of sides s and $s\sqrt{2}$ inscribed in an isosceles triangle of base 2 and height 3, where s is the side-length of the cube. (The $s\sqrt{2}$ side of the rectangle lies on the base of the triangle.) Similar triangles yield $s/3 = (1 - s\sqrt{2}/2)/1$, or $s = (9\sqrt{2} - 6)/7$.

2. Let $A_0, A_1, \ldots, A_{n-1}$ be the vertices of a regular n-gon inscribed in the unit circle. Prove that

   $$A_0A_1 \cdot A_0A_2 \cdots A_0A_{n-1} = n.$$

   Solution: In a system of complex coordinates, place each vertex A_k ε^k, where $\varepsilon = e^{2i\pi/n}$. Then

   $$A_0A_1 \cdot A_0A_2 \cdots A_0A_{n-1} = |(1-\varepsilon)(1-\varepsilon^2)\cdots(1-\varepsilon^{n-1})|.$$

   Observe that, in general,

   $$\begin{aligned}(z-\varepsilon)(z-\varepsilon^2)\cdots(z-\varepsilon^{n-1}) &= \frac{1}{z-1}(z-1)(z-\varepsilon)\cdots(z-\varepsilon^{n-1}) \\ &= \frac{z^n-1}{z-1} \\ &= z^{n-1} + z^{n-2} + \cdots + 1\end{aligned}$$

   By continuity, this equality also holds for $z = 1$. Hence

   $$A_0A_1 \cdot A_0A_2 \cdots A_0A_{n-1} = 1^{n-1} + 1^{n-2} + \cdots + 1 = n,$$

   and the identity is proved.

3. A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC?

   Solution (Putnam 1997 A-1): The centroid G of the triangle is collinear with H and O (Euler line), and the centroid lies two-thirds of the way from A to M. Therefore H is also two-thirds of the way from A to F, so $AF = 15$. Since the triangles BFH and AFC are similar (they're right triangles and

   $$\angle HBC = \pi/2 - \angle C = \angle CAF),$$

   we have

   $$BF/FH = AF/FC$$

   or

   $$BF \cdot FC = FH \cdot AF = 75.$$

   Now

   $$BC^2 = (BF + FC)^2 = (BF - FC)^2 + 4BF \cdot FC,$$

   but

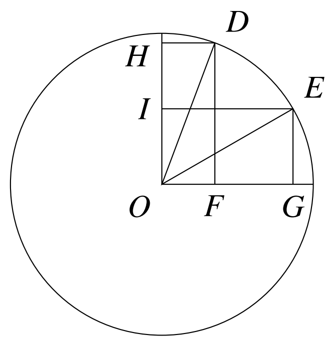
   $$BF - FC = BM + MF - (MC - MF) = 2MF = 22,$$

   so

   $$BC = \sqrt{22^2 + 4 \cdot 75} = \sqrt{784} = 28.$$

4. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position, of s.

   Solution (Putnam 1998 A-2): First solution: to fix notation, let A be the area of region $DEFG$, and B be the area of $DEIH$; further let C denote the area of sector ODE, which only depends on the arc length of s. If $[XYZ]$ denotes the area of triangle $[XYZ]$, then we have $A = C + [OEG] - [ODF]$ and $B = C + [ODH] - [OEI]$. But clearly $[OEG] = [OEI]$ and $[ODF] = [ODH]$, and so $A + B = 2C$.



Second solution: We may parametrize a point in s by any of x, y, or $\theta = \tan^{-1}(y/x)$. Then A and B are just the integrals of $y\,dx$ and $x\,dy$ over the appropriate intervals; thus $A+B$ is the integral of $x\,dy - y\,dx$ (minus because the limits of integration are reversed). But $d\theta = x\,dy - y\,dx$, and so $A+B = \Delta\theta$ is precisely the radian measure of s. (Of course, one can perfectly well do this problem by computing the two integrals separately. But what's the fun in that?)

5. Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta\to 0} |EF|$.

   Solution (Putnam 1999 B-1): The answer is 1/3. Let G be the point obtained by reflecting C about the line AB. Since $\angle ADC = \frac{\pi-\theta}{2}$, we find that $\angle BDE = \pi - \theta - \angle ADC = \frac{\pi-\theta}{2} = \angle ADC = \pi - \angle BDC = \pi - \angle BDG$, so that E, D, G are collinear. It follows that $\angle BGE = \frac{\theta}{2}$ and $\angle BEG = \frac{3\theta}{2}$. Hence

$$|EF| = \frac{|BE|}{|BC|} = \frac{|BE|}{|BG|} = \frac{\sin(\theta/2)}{\sin(3\theta/2)},$$

   where we have used the law of sines in $\triangle BDG$. But by l'Hôpital's Rule,

$$\lim_{\theta\to 0} \frac{\sin(\theta/2)}{\sin(3\theta/2)} = \lim_{\theta\to 0} \frac{\cos(\theta/2)}{3\cos(3\theta/2)} = 1/3.$$