2021 - ISU Putnam Practice Set 8 - Solutions

Friday, November 11, 2022

Number Theory II

1. Solve in positive integers the equation

$$2^x \cdot 3^y = 1 + 5^z.$$

Solution: Reducing modulo 4, the right-hand side of the equation becomes equal to 2. So the left-hand side is not divisible by 4, which means that $x = 1$. If $y > 1$, then reducing modulo 9 we find that z has to be divisible by 6. A reduction modulo 6 makes the left-hand side 0, while the right-hand side would be $1 + (-1)^z = 2$. This cannot happen. Therefore, $y = 1$, and we obtain the unique solution $x = y = z = 1$.

2. Let (x_n) be a sequence of positive integers satisfying the recurrence relation $x_{n+1} = 5x_n - 6x_{n-1}$. Prove that infinitely many terms of the sequence are composite.

Solution: The recurrence relation is linear. Using the characteristic equation we find that

$$x_n = A2^n + B3^n,$$

where $A = 3x_0 - x_1$ and $B = x_1 - 2x_0$. We see that A and B are integers. Now let us assume that all but finitely many terms of the sequence are prime. Then $A, B \neq 0$, and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} 3^n \left(A \left(\frac{2}{3} \right)^n + B \right) = \infty.$$

Let n be sufficiently large so that x_n is a prime number different from 2 and 3. Then for $k \geq 1$,

$$x_{n+k(p-1)} = A\Delta 2^{n+k(p-1)} + B\Delta 3^{n+k(p-1)} = A\Delta 2^n \Delta (2^{p-1})^k + B\Delta 3^n \Delta (3^{p-1})^k.$$

By Fermat's little theorem, this is congruent to $A\Delta 2^n + B\Delta 3^n$ modulo p, hence to x_n which is divisible by p. So the terms of the subsequence $x_{n+k(p-1)}$, $k \geq 1$, are divisible by p, and increase to infinity. This can happen only if the terms become composite at some point, which contradicts our assumption. The problem is solved.

3. How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

   Solution Putnam 1989 A1: 101.

   Let k_n represent the member of the sequence with n 1's. It is obvious that 101 divides k_{2n}. So we need only consider k_{2n+1}.

   But

$$k_{2n+1} = 1 + 10^2 + 10^4 + \ldots + 10^{4n} = (10^{4n+2} - 1)/99 = \frac{10^{2n+1} + 1}{11} \frac{10^{2n+1} - 1}{9}.$$

   Each of these is integral: the first is $1 - 10 + 102 - \ldots + 10^{2n}$, the second is $11 \ldots 1$ $(2n + 11's)$.

4. Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

   Solution Putnam 2018 A1: By clearing denominators and regrouping, we see that the given equation is equivalent to

$$(3a - 2018)(3b - 2018) = 2018^2.$$

   Each of the factors is congruent to $1 \pmod 3$. There are 6 positive factors of $2018^2 = 2^2 \cdot 1009^2$ that are congruent to $1 \pmod 3$: 1, 2^2, 1009, $2^2 \cdot 1009$, 1009^2, $2^2 \cdot 1009^2$. These lead to the 6 possible pairs: $(a, b) = (673, 1358114)$, $(674, 340033)$, $(1009, 2018)$, $(2018, 1009)$, $(340033, 674)$, and $(1358114, 673)$.

   As for negative factors, the ones that are congruent to $1 \pmod 3$ are $-2, -2 \cdot 1009, -2 \cdot 1009^2$. However, all of these lead to pairs where $a \leq 0$ or $b \leq 0$.

5. Find all right triangles whose sides are positive integers and whose perimeter is numerically equal to their area.

   Solution: We use the characterization of Pythagorean triples as

$$a = k(u^2 - v^2), b = 2kuv, c = k(u^2 + v^2)$$

   for some k, u, v positive integers, $gcd(u, v) = 1$. The condition from the statement translates into

$$k^2uv(u^2 - v^2) = 2k(u^2 + uv).$$

Dividing by ku and moving 2v to the left we obtain

$$v(ku^2 - kv^2 - 2) = 2u.$$

Since $gcd(u,v) = 1$, $v = 1$ or 2. If $v = 1$ we obtain $ku^2 - k - 2 = 2u$, that is $ku(u-2) = k-2$. This can only happen for small values of k and u, and an easy check yields $k = u = 2$. We then obtain the Pythagorean triple (6, 8, 10).

If $v = 2$, then $ku(u?1) = 4k+2$, and again this can only happen for small values of k and u. An easy check yields $k = 1$, $u = 3$, and we obtain the Pythagorean triple (5, 12, 13).

BETTER SOLUTION DUE TO ANDREW OSBORNE 11/11/2022: Let a, b, c be the lengths of the sides of the right triangle, c being the hypotenuse. Then

$$\begin{aligned}
&\frac{ab}{2} = a+b+c = a+b+\sqrt{a^2+b^2}\\
&\Rightarrow \left(\frac{ab}{2} - (a+b)\right)^2 = a^2+b^2\\
&\Rightarrow \frac{a^2b^2}{4} - a^2b - ab^2 + a^2 + 2ab + b^2 = a^2 + b^2\\
&\Rightarrow ab\left(\frac{ab}{4} - a - b - 2\right) = 0\\
&\Rightarrow b - 2 = a\left(\frac{b}{4} - 1\right)\\
&\Rightarrow a = \frac{b-2}{b/4-1} = \frac{4b-8}{b-4} = 4 + \frac{8}{b-4}.
\end{aligned}$$

Thus $b-4$ is an integer dividing 8 and there are 8 cases to consider: $b-4 = \pm1, \pm2, \pm4, \pm8$ yielding $b = -4, 0, 2, 3, 5, 6, 8, 12$. The first two are nonpositive and discarded. The second two yield nonpositive a and are discarded. $b = 5$ yields $a = 12$ and $c = 13$. Symmetrically, $b = 12$ yields $(a, c) = (5, 13)$. Similarly $b = 6$ or $b = 8$ yield the triple $(6, 8, 10)$. Thus the two triples are $(5, 12, 13)$ and $(6, 8, 10)$.