2021 - ISU Putnam Practice Set 8 - Solutions

Wednesday, November 3, 2021

Pigeonhole Principle

1. Inside a circle of radius 4 are chosen 61 points. Show that among them there are 2 points at a distance at most $\sqrt{2}$ from each other.

   Solution: Place the circle at the origin of the coordinate plane and consider the rectangular grid determined by points of integer coordinates. The circle is inscribed in an 8×8 square decomposed into 64 unit squares. Because $3^2 + 3^2 > 4^2$, the four unit squares at the corners lie outside the circle. The interior of the circle is therefore covered by 60 squares, which are our 'holes'. The 61 points are the 'pigeons', and by the pigeonhole principle two lie inside the same square. The distance between them does not exceed the length of the diagonal, which is 2. The problem is solved.

2. Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

   Solution: (2002 A2) Take the great circle determined by two of the points. The remaining three points each lie in at least one of the closed hemispheres determined by this great circle. By the Pigeonhole Principle, at least 2 of them lie in one of the closed hemispheres. Those 2, along with the 2 on the boundary, give us 4 points.

3. Let A be any set of 20 distinct integers chosen from the arithmetic progression $\{1,4,7,\ldots,100\}$. Prove that there must be two distinct integers in A whose sum is 104.

   Solution: (1978 A1) The given set can be divided into 18 subsets $\{1\}$, $\{4,100\}$, $\{7,97\}$, $\{10,94\}$,. . . , $\{49,55\}$, $\{52\}$. By the pigeonhole principle two of the numbers will be in the same set, and all 2-element subsets shown verify that the sum of their elements is 104.

4. During a month with 30 days a baseball team plays at least a game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

   Solution: Let a_j the number of games played from the 1st through the jth day of the month. Then $a_1, a_2, \ldots, a_30$ is an increasing sequence of distinct positive integers, with $1 \le a_j \le 45$.

Likewise, $b_j = a_j + 14,\ j = 1, \ldots, 30$ is also an increasing sequence of distinct positive integers with $15 \le b_j \le 59$. The 60 positive integers $a_1, \ldots, a_30, b_1, ..., b_30$ are all less than or equal to 59, so by the pigeonhole principle two of them must be equal. Since the a_j's are all distinct integers, and so are the b_j's, there must be indices i and j such that $a_i = b_j = a_j + 14$. Hence $a_i - a_j = 14$, i.e., exactly 14 games were played from day $j + 1$ through day i.

5. The points of the plane are colored by finitely many colors. Prove that one can find a rectangle with vertices of the same color.

   Solution: Let there be p colors. Consider a $(p+1) \times \left(\binom{p+1}{2} + 1\right)$ rectangular grid. By the pigeonhole principle, each of the $\binom{p+1}{2} + 1$ horizontal segments contains two points of the same color. There are $\binom{p+1}{2}$ possible configurations of monochromatic pairs, so two must repeat. The repeating pairs are vertices of a monochromatic rectangle.