Linear Algebra

1. Let $A$ and $B$ be $2 \times 2$ matrices with real entries satisfying $(AB - BA)^n = I_2$ for some positive integer $n$. Prove that $n$ is even and $(AB - BA)^4 = I_2$.

**Solution:** To simplify our work, we note that in general, for any two square matrices $A$ and $B$ of arbitrary dimension, the trace of $AB - BA$ is zero. We can therefore write

$$AB - BA = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$\[\text{But then } (AB - BA)^2 = kI_2, \text{ where } k = a^2 + bc. \text{ This immediately shows that an odd power of } AB - BA \text{ is equal to a multiple of this matrix. The odd power cannot equal } I_2 \text{ since it has trace zero. Therefore, } n \text{ is even.} \]

The condition from the statement implies that $k$ is a root of unity. But there are only two real roots of unity and these are $1$ and $-1$. The squares of both are equal to 1. It follows that $(AB - BA)^4 = k^2I_2 = I_2$, and the problem is solved.

2. Let $A$ and $B$ be $3 \times 3$ matrices with real elements such that

$$\det A = \det B = \det(A + B) = \det(A - B) = 0.$$

Prove that $\det(xA + yB) = 0$ for any real numbers $x$ and $y$.

**Solution:** Expand the determinant as

$$\det(xA + yB) = a_0(x)y^3 + a_1(x)y^2 + a_2(x)y + a_3(x),$$

where $a_i(x)$ are polynomials of degree at most $i$, $i = 0, 1, 2, 3$. For $y = 0$ this gives $\det(xA) = x^3 \det A = 0$, and hence $a_3(x) = 0$ for all $x$. Similarly, setting $y = x$ we obtain $\det(xA + xB) = x^3 \det(A + B) = 0$, and thus

$$a_0(x)x^3 + a_1(x)x^2 + a_2(x)x = 0.$$
Also, for \( y = -x \) we obtain \( \det(xA - xB) = x^3 \det(A - B) = 0 \); thus
\[-a_0(x)x^3 + a_1(x)x^2 - a_2(x)x = 0.\]

Adding these two relations gives \( a_1(x) = 0 \) for all \( x \). For \( x = 0 \) we find that \( \det(yB) = y^3 \det B = 0 \), and hence
\[a_0(0)y^3 + a_2(0)y = 0\]
for all \( y \). Therefore, \( a_0(0) = 0 \). But \( a_0(x) \) is constant, so \( a_0(x) = 0 \). This implies \( a_2(x)x = 0 \) for all \( x \), and so \( a_2(x) = 0 \) for all \( x \). We conclude that \( \det(xA + yB) \) is identically equal to zero, and the problem is solved.

3. Let \( a, b, c, d \) be positive numbers different from 1, and \( x, y, z, t \) real numbers satisfying
\[a^x = bcd, \quad b^y = cda, \quad c^z = dab, \quad d^t = abc.\]
Prove that
\[
\begin{vmatrix}
-x & 1 & 1 & 1 \\
1 & -y & 1 & 1 \\
1 & 1 & -z & 1 \\
1 & 1 & 1 & -t \\
\end{vmatrix}
= 0.
\]

**Solution:** Taking the logarithms of the four relations from the statement, we obtain the following system of linear equations in the unknowns \( \ln a, \ln b, \ln c, \ln d \):

\[-x \ln a + \ln b + \ln c + \ln d = 0,\]
\[\ln a - y \ln b + \ln c + \ln d = 0,\]
\[\ln a + \ln b - z \ln c + \ln d = 0,\]
\[\ln a + \ln b + \ln c - t \ln d = 0.\]

We are given that this system has a nontrivial solution. Hence the determinant of the coefficient matrix is zero, which is what had to be proved.

4. \( M \) and \( N \) are real unequal \( n \times n \) matrices satisfying \( M^3 = N^3 \) and \( M^2N = N^2M \). Can we choose \( M \) and \( N \) so that \( M^2 + N^2 \) is invertible?

**Solution (Putnam 1991 A-2):** No.

\[(M^2 + N^2)M = M^3 + N^2M = N^3 + M^2N = (M^2 + N^2)N.\]
But now if \( M^2 + N^2 \) was invertible, we could multiply by its inverse to get \( M = N \), whereas we are told \( M \) and \( N \) are unequal.

5. Let \( A \) and \( B \) be \( 2 \times 2 \) matrices with integer entries such that \( A, A + B, A + 2B, A + 3B, \) and \( A + 4B \) are all invertible matrices whose inverses have integer entries. Show that \( A + 5B \) is invertible and that its inverse has integer entries.

Solution (Putnam 1994 A-4): Let \( A = (a_{ij}) \), \( B = (b_{ij}) \). Given any matrix \( C = (c_{ij}) \), \( C \) is invertible iff the determinant \( d = c_{11}c_{22} - c_{12}c_{21} \) is non-zero and the inverse is then \((d_{ij})\) where \( d_{11} = c_{22}/d \), \( d_{12} = -c_{12}/d \), \( d_{21} = -c_{21}/d \), \( d_{22} = c_{11}/d \). Thus if the inverse has integer entries, then \( d \) divides all the entries in the original matrix and hence \( d^2 \) divides \( d \), so \( d = 1 \) or \(-1\).

Multiplying out, we find that \( \det(A + nB) = a + nh + n^2k \), where \( a = \det A = 1 \) or \(-1\), \( h = (a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11}) \), and \( k = \det B \). Since \( \det(A + nB) \) is integral with integral inverses, \( \det(A + nB) = \pm 1 \) for \( n = 0, 1, 2, 3, 4 \). By the Pigeon-hole principle, \( \det(A + nB) \) must take one of these two values at least 3 times, which is impossible for a quadratic polynomial in \( n \) unless it is constant. In particular, \( \det(A + 5B) = \det A = \pm 1 \) and so is invertible with integral inverse.