2023 - ISU Putnam Practice Set 7 - Solutions

Thursday, October 19, 2023

Linear Algebra

1. Let A and B be 2×2 matrices with real entries satisfying $(AB - BA)^n = I_2$ for some positive integer n. Prove that n is even and $(AB - BA)^4 = I_2$.

   Solution: To simplify our work, we note that in general, for any two square matrices A and B of arbitrary dimension, the trace of $AB - BA$ is zero. We can therefore write

   $$AB - BA = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

   But then $(AB - BA)^2 = kI_2$, where $k = a^2 + bc$. This immediately shows that an odd power of $AB - BA$ is equal to a multiple of this matrix. The odd power cannot equal I_2 since it has trace zero. Therefore, n is even.

   The condition from the statement implies that k is a root of unity. But there are only two real roots of unity and these are 1 and -1. The squares of both are equal to 1. It follows that $(AB - BA)^4 = k^2 I_2 = I_2$, and the problem is solved.

2. Let A and B be 3×3 matrices with real elements such that

   $$\det A = \det B = \det(A + B) = \det(A - B) = 0.$$

   Prove that $\det(xA + yB) = 0$ for any real numbers x and y.

   Solution: Expand the determinant as

   $$\det(xA + yB) = a_0(x)y^3 + a_1(x)y^2 + a_2(x)y + a_3(x),$$

   where $a_i(x)$ are polynomials of degree at most i, $i = 0, 1, 2, 3$. For $y = 0$ this gives $\det(xA) = x^3 \det A = 0$, and hence $a_3(x) = 0$ for all x. Similarly, setting $y = x$ we obtain $\det(xA + xB) = x^3 \det(A + B) = 0$, and thus

   $$a_0(x)x^3 + a_1(x)x^2 + a_2(x)x = 0.$$

Also, for $y = -x$ we obtain $\det(xA - xB) = x^3 \det(A - B) = 0$; thus

$$-a_0(x)x^3 + a_1(x)x^2 - a_2(x)x = 0.$$

Adding these two relations gives $a_1(x) = 0$ for all x. For $x = 0$ we find that $\det(yB) = y^3 \det B = 0$, and hence

$$a_0(0)y^3 + a_2(0)y = 0$$

for all y. Therefore, $a_0(0) = 0$. But $a_0(x)$ is constant, so $a_0(x) = 0$. This implies $a_2(x)x = 0$ for all x, and so $a_2(x) = 0$ for all x. We conclude that $\det(xA + yB)$ is identically equal to zero, and the problem is solved.

3. Let a, b, c, d be positive numbers different from 1, and x, y, z, t real numbers satisfying $a^x = bcd$, $b^y = cda$, $c^z = dab$, $d^t = abc$. Prove that

$$\det \begin{pmatrix} -x & 1 & 1 & 1 \\ 1 & -y & 1 & 1 \\ 1 & 1 & -z & 1 \\ 1 & 1 & 1 & -t \end{pmatrix} = 0.$$

Solution: Taking the logarithms of the four relations from the statement, we obtain the following system of linear equations in the unknowns $\ln a$, $\ln b$, $\ln c$, $\ln d$:

$$\begin{aligned} -x\ln a + \ln b + \ln c + \ln d &= 0, \\ \ln a - y\ln b + \ln c + \ln d &= 0, \\ \ln a + \ln b - z\ln c + \ln d &= 0, \\ \ln a + \ln b + \ln c - t\ln d &= 0. \end{aligned}$$

We are given that this system has a nontrivial solution. Hence the determinant of the coefficient matrix is zero, which is what had to be proved.

4. M and N are real unequal $n \times n$ matrices satisfying $M^3 = N^3$ and $M^2N = N^2M$. Can we choose M and N so that $M^2 + N^2$ is invertible?

Solution (Putnam 1991 A-2): No.

$$(M^2 + N^2)M = M^3 + N^2M = N^3 + M^2N = (M^2 + N^2)N.$$

But now if M^2+N^2 was invertible, we could multiply by its inverse to get $M=N$, whereas we are told M and N are unequal.

5. Let A and B be 2×2 matrices with integer entries such that A, $A+B$, $A+2B$, $A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

   Solution (Putnam 1994 A-4): Let $A=(a_{ij})$, $B=(b_{ij})$. Given any matrix $C=(c_{ij})$, C is invertible iff the determinant $d=c_{11}c_{22}-c_{12}c_{21}$ is non-zero and the inverse is then (d_{ij}) where $d_{11}=c_{22}/d$, $d_{12}=-c_{12}/d$, $d_{21}=-c_{21}/d$, $d_{22}=c_{11}/d$. Thus if the inverse has integer entries, then d divides all the entries in the original matrix and hence d^2 divides d, so $d=1$ or -1.

   Multiplying out, we find that $\det(A+nB)=a+nh+n^2k$, where $a=\det A=1$ or -1, $h=(a_{11}b_{22}-a_{12}b_{21}-a_{21}b_{12}+a_{22}b_{11})$, and $k=\det B$. Since $\det(A+nB)$ is integral with integral inverses, $\det(A+nB)=\pm 1$ for $n=0,1,2,3,4$. By the Pigeon-hole principle, $\det(A+nB)$ must take one of these two values at least 3 times, which is impossible for a quadratic polynomial in n unless it is constant. In particular, $\det(A+5B)=\det A=\pm 1$ and so is invertible with integral inverse.