2021 - ISU Putnam Practice Set 7 - Solutions

Friday, November 4, 2022

Number Theory

1. How many positive integers n are there such that n is an exact divisor of at least one of the numbers $10^{40}, 20^{30}$.

   Solution Putnam 1983 A1: For $m \in \mathbb{Z}_+$, let $\tau(m)$ be the number of d in $\mathbb{Z}_+$ such that $d|m$. The number of n such that $n|a$ or $n|b$ is

   $$\tau(a)+\tau(b)-\tau(gcd(a,b)).$$

   Also $\tau(p^s q^t) = (s+1)(t+1)$ for $p,q,s,t \in \mathbb{Z}_+$ with p,q distinct primes. Thus the desired count is

   $$\tau(2^{40}\cdot 5^{40}) = \tau(2^{40}\cdot 5^{40})+\tau(2^{60}\cdot 5^{30})-\tau(2^{40}\cdot 5^{30}) = 41^2+61\cdot 31-41\cdot 31 = 1681+620 = 2301.$$

2. Find the integers n for which $(n^3-3n^2+4)/(2n-1)$ is an integer.

   Solution: If we multiplied the fraction by 8, we would still get an integer. Note that

   $$8\frac{n^3-3n^2+4}{2n-1} = 4n^2-10n-5+\frac{27}{2n-1}.$$

   Hence $2n-1$ must divide 27. This happens only when $2n-1 = \pm 1, \pm 3, \pm 9, \pm 27$, that is, when $n = -13, -4, -1, 1, 2, 5, 14$. An easy check shows that for each of these numbers the original fraction is an integer.

3. Let n be an integer greater than 2. Prove that $n(n-1)^4+1$ is the product of two integers greater than 1.

   Solution : The polynomial

   $$P(n) = n(n-1)^4+1 = n^5-4n^4+6n^3-4n^2+n+1$$

   does not have integer zeros, so we should be able to factor it as a product a quadratic and a cubic polynomial. This means that

   $$P(n) = (n^2+an+1)(n^2+bn^2+cn+1),$$

for some integers a, b, c. Identifying coefficients, we must have

$$\begin{aligned} a+b &= -4 \\ c+ab+1 &= 6 \\ b+ac+1 &= -4 \\ a+c &= 1. \end{aligned}$$

rom the first and last equations, we obtain $b-c=-5$, and from the second and the third, $(b-c)(a-1)=10$. It follows that $a-1=-2$; hence $a=-1$, $b=-4+1=-3$, $c=1+1=2$. Therefore,

$$n(n-1)^4+1=(n^2-n+1)(n^3-3n^2+2n+1),$$

a product of integers greater than 1.

4. Solve in positive integers the equation

$$x^{x+y}=y^{y-x}.$$

Solution: The numbers x and y have the same prime factors

$$x=\prod_{i=1}^{k} p_i^{\alpha_i}, \qquad y=\prod_{i=1}^{k} p_i^{\beta_i}.$$

The equality from the statement can be written as

$$\prod_{i=1}^{k} p_i^{\alpha_i(x+y)}=\prod_{i=1}^{k} p_i^{\beta_i(y-x)};$$

hence $\alpha_i(y+x)=\beta_i(y-x)$ for $i=1,\ldots,k$. From here we deduce that $\alpha_i<\beta_i$ for $i=1,2,\ldots,k$ and therefore x divides y. Writing $y=zx$, the equation becomes

$$x^{x(z+1)}=(xz)^{x(z-1)},$$

which implies

$$x^2=z^{z-1}$$

and then

$$y^2=(xz)^2=z^{z+1}.$$

A power is a perfect square if either the base is itself a perfect square or if the exponent is even. For $z=t^2$, $t\geq 1$, we have $x=t^{t^2-1}, y=t^{t^2+1}$, which is one family of solutions. For $z-1=2s$, $s\geq 0$, we obtain the second family of solutions $x=(2s+1)^s$, $y=(2s+1)^s+1$.

5. Show that each positive integer can be written as the difference of two positive integers having the same number of prime factors.

   Solution: If n is even, then we can write it as $(2n) - n$. If n is odd, let p be the smallest odd prime that does not divide n. Then write $n = (pn) - ((p-1)n)$. The number pn contains exactly one more prime factor than n. As for $(p-1)n$, it is divisible by 2 because $p-1$ is even, while its odd factors are less than p, so they all divide n by the minimality of p. Therefore, $(p-1)n$ also contains exactly one more prime factor than n, and therefore pn and $(p-1)n$ have the same number of prime factors.