# 2021 - ISU Putnam Practice Set 7 - Solutions

## Wednesday, October 20, 2021

## Polynomials

1. Let $P(x)$ be a polynomial with complex coefficients. Prove that $P(x)$ is an even func- tion if and only if there exists a polynomial $Q(x)$ with complex coefficients satisfying $P(x) = Q(x)Q(-x)$. (Sorry for the repeated problem.)

   **Solution:** If such a $Q(x)$ exists, it is clear that $P(x)$ is even. Conversely, assume that $P(x)$ is an even function. Writing $P(x) = P(?x)$ and identifying coefficients, we conclude that no odd powers appear in $P(x)$. Hence

   $$P(x) = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdots + a_2x^2 + a_0 = P_1(x^2).$$

   Factoring

   $$P_1(y) = a(y - y1)(y - y2)\cdots(y - yn),$$

   we have

   $$P(x) = a(x^2 - y_1)(x^2 - y_2)\cdots(x^2 - y_n).$$

   Now choose complex numbers $b, x_1, x_2, \cdots, x_n$ such that $b^2 = (-1)^n a$ and $x_j^2 = y_j,\ j = 1, 2, \ldots, n$. We have the factorization

   $$\begin{aligned} P(x) &= b^2(x_1^2 - x^2)(x_2^2 - x^2)\cdots(x_n^2 - x^2) \\ &= b^2(x_1 - x)(x_1 + x)(x_2 - x)(x_2 + x)\cdots(x_n - x)(x_n + x) \\ &= [b(x_1 - x)(x_2 - x)\cdots(x_n - x)][b(x_1 + x)(x_2 + x)\cdots(x_n + x)] = Q(x)Q(-x), \end{aligned}$$

   where $Q(x) = b(x_1 - x)(x_2 - x)\cdots(x_n - x)$. This completes the proof.

2. Suppose $p(x) = x^4 + ax^3 + bx^2 + cx + d$ be a polynomial with rational coefficients. Suppose $p(x)$ has exactly one real root $r$. Prove that $r$ is rational.

   **Solution:** Since nonreal roots occur in complex conjugate pairs, $p(x)$ must have an even number of real roots, counting multiplicity. Therefore $r$ is either a double or quadruple root. If $r$ is a quadruple root, then $p(x) = (x - r)^4$ and $r = -a/4$ is rational. Suppose otherwise that $p(x) = (x - r)^2 q(x)$, where $q(x)$ is an irreducible quadratic. The derivative $p'(x)$ is equal to $(x-r)f(x)$, where $f(x) = 2q(x) + (x-r)q'(x)$. Since $q(r)$ does not vanish, $f(r) \neq q(r)$ so that $f(x)$ and $q(x)$ are distinct and relatively prime. The monic gcd of $p(x)$ and $p'(x)$ must therefore be $x-r$. Since (by the Euclidean algorithm) this is a polynomial with rational coefficients, therefore $r$ is rational.

3. Let $a \in \mathbb{C}$ and $n \geq 2$. Prove that the polynomial equation $ax^n + x + 1 = 0$ has a root of absolute value less than or equal to 2.

   **Solution:** Change variables to $z = \frac{1}{x}$ to obtain the polynomail $Q(z) = z^n + z^{n-1} + a$. If all zeros of $ax^n + x + 1$ were outside the circle of radius 2 centered at the origin, then all zeroes of $Q(z)$ would lie in the interior of the circle of radius $\frac{1}{2}$. By Lucas' Theorem, the same is true of $Q'(z) = nz^{n-1} + (n-1)z^{n-2}$, which has $z = \frac{n-1}{n} \geq \frac{1}{2}$ as a root - a contradiction.

4. Suppose $u, v, w, z$ are complex numbers for which $u + v + w + z = u^2 + v^2 + w^2 + z^2 = 0$. Prove that

$$(u^4 + v^4 + w^4 + z^4)^2 = 4(u^8 + v^8 + w^8 + z^8).$$

   **Solution:** Let $f(t) = (t-u)(t-v)(t-w)t-z)$. Since $u + v + w + z = 0$ and

$$uv + uw + uz + vw + vz + wz = \frac{1}{2}\big(((u+v+w+z)^2 - (u^2+v^2+w^2+z^2)\big) = 0,$$

   we must have $f(t) = t^4 - at - b$ for some $a, b \in \mathbb{C}$. Set $s_n = u^n + v^n + w^n + z^n$ for $n \geq 1$. Then

$$s_4 = u^4 + v^4 + w^4 + z^4 = a(u+v+w+z) + 4b = 4b.$$

   For $k \geq 1$,

$$\begin{aligned} s_{k+4} &= u^{k+4} + v^{k+4} + w^{k+4} + z^{k+4} \\ &= (au+b)u^k + (av+b)v^k + (aw+b)w^k + (az+b)z^k \\ &= as_{k+1} + bs_k. \end{aligned}$$

   Hence $s_5 = 0$ and $s_8 = as_5 + bs_4 = 4b^2$, so that $4s_8 = 16b^2 = s_4^2$, as desired.

5. Let $k$ be the smallest positive integer for which there exist distinct integers $m_1, m_2, m_3, m_4, m_5$ such that the polynomial

$$p(x) = (x-m_1)(x-m_2)(x-m_3)(x-m_4)(x-m_5)$$

has exactly $k$ nonzero coefficients. Find, with proof, a set of integers $m_1, m_2, m_3, m_4, m_5$ for which this minimum $k$ is achieved.

**Solution: (1985 B1)** 3. For example: $0, \pm 1, \pm 2$. The example given has three non-zero coefficients: $x(x^2-1)(x^2-4) = x^5 - 5x^3 + 4x$. So we have to show that two nonzero coefficients are not possible. The polynomial cannot be $x^5 + ax^n$ with $n > 1$, because it would then have 0 as a repeated root and hence less than 5 distinct roots. $x^5 + ax$ has at most 3 real roots, and $x^5 + a$ has only one real root.