2023 - ISU Putnam Practice Set 6 - Solutions

Thursday, October 12, 2023

Trigonometry

1. Show that the trigonometric equation

$$\sin(\cos x) = \cos(\sin x)$$

has no solutions.

Solution: Because $-\frac{\pi}{2} < 1 < \sin(x) < 1 < \frac{\pi}{2}$, $\cos(\sin(x)) > 0$. Hence $\sin(\cos x)) > 0$, and so $\cos(x) > 0$. We deduce that the only possible solutions can lie in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Note that if x is a solution, then $-x$ is also a solution; thus we can restrict our attention to the first quadrant. Rewrite the equation as

$$\sin(\cos x)) = \sin(\frac{\pi}{2} - \sin(x)).$$

Then $\cos x = \frac{\pi}{2} - \sin(x)$, and so $\sin x + \cos x = \frac{\pi}{2}$. This equality cannot hold, since the range of the function $f(x) = \sin(x) + \cos(x) = \sqrt{2}\cos(\frac{\pi}{4} - x)$ is $[-\sqrt{2}, \sqrt{2}]$ and $\frac{\pi}{2} > \sqrt{2}$.

2. Prove that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin 1^\circ}.$$

Solution: Multiply the left-hand side by $\sin 1^\circ$ and transform it using the identity

$$\frac{\sin((k+1)^\circ - k^\circ)}{\sin k^\circ \sin(k+1)^\circ} = \cot k^\circ - \cot(k+1)^\circ.$$

We obtain

$$\cot 45^\circ - \cot 46^\circ + cot 47^\circ - cot 48^\circ + \cdots + \cot 131^\circ - \cot 132^\circ + \cot 133^\circ - \cot 134^\circ$$
$$= \cot 45^\circ - (\cot 46^\circ + \cot 134^\circ) + (\cot 47^\circ + \cot 133^\circ) - \cdots + (\cot 89^\circ + \cot 91^\circ) - \cot 90^\circ$$
$$= \cot 45^\circ - \cot 90^\circ = 1 - 0 = 1.$$

3. Solve the following system of equations in real numbers:

$$\frac{3x-y}{x-3y} = x^2,$$
$$\frac{3y-z}{y-3z} = y^2,$$
$$\frac{3z-x}{z-3x} = z^2.$$

Solution: From the first equation, it follows that if x is 0, then so is y, making x^2 indeterminate; hence x, and similarly y and z, cannot be 0. Solving the equations, respectively, for y, z, and x, we obtain the equivalent system

$$y = \frac{3x-x^3}{1-3x^2},$$
$$z = \frac{3y-y^3}{1-3y^2},$$
$$x = \frac{3z-z^3}{1-3z^2},$$

where x, y, z are real nonzero numbers.

There exists a unique u in the interval $(-\pi/2, \pi/2)$ such that $x = \tan u$ Then

$$y = \frac{3\tan u - \tan^3 u}{1-3\tan^2 u} = \tan(3u),$$
$$z = \frac{3\tan(3u) - \tan^3(3u)}{1-3\tan^2(3u)} = \tan(9u),$$
$$x = \frac{3\tan(9u) - \tan^3(9u)}{1-3\tan^2(9u)} = \tan(27u).$$

So $\tan u = \tan(27u)$ and hence u and $27u$ differ by an integer multiple of π. Therefore $u = \frac{k\pi}{26}$ for some k satisfying $\frac{-\pi}{2} < \frac{k\pi}{26} < \frac{\pi}{2}$. Thus $k = \pm 1, \pm 2, \ldots, \pm 12$, each generating a corresponding triple

$$x = \tan\frac{k\pi}{26}, \qquad y = \tan\frac{3k\pi}{26}, \qquad z = \tan\frac{9k\pi}{26}.$$

Each such triple solves the original system of equations.

4. An ellipse, whose semi-axes have lengths a and b, rolls without slipping on the curve $y = c\sin\left(\frac{x}{a}\right)$. How are a, b, c related, given that the ellipse completes one revolution when it traverses one period of the curve?

Solution (Putnam 1995 B-2): For those who haven't taken enough physics, "rolling without slipping" means that the perimeter of the ellipse and the curve pass at the same rate, so all we're saying is that the perimeter of the ellipse equals the length of one period of the sine curve. So set up the integrals:

$$\int_0^{2\pi} \sqrt{(-a\sin\theta)^2 + (b\cos\theta)^2}\, d\theta = \int_0^{2\pi a} \sqrt{1 + (c/a\cos x/a)^2}\, dx.$$

Let $\theta = x/a$ in the second integral and write 1 as $\sin^2\theta + \cos^2\theta$ and you get

$$\int_0^{2\pi} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{a^2\sin^2\theta + (a^2+c^2)\cos^2\theta}\, d\theta.$$

Since the left side is increasing as a function of b, we have equality if and only if $b^2 = a^2 + c^2$.

5. Compute the indefinite integral

$$\int \sqrt{\frac{1-x}{1+x}} dx, \qquad x \in (-1,1).$$

Solution: If we substitute $x = sint$, then $dx = \cos t dt$ and the integral becomes

$$\begin{aligned}
\int \sqrt{\frac{1-\sin t}{1+\sin t}} \cos t dt &= \int \sqrt{\frac{1-\sin^2 t}{(1+\sin t)^2}} \cos t dt \\
&= \int \sqrt{\frac{\cos^2 t}{(1+\sin t)^2}} \cos t dt \\
&= \int \frac{\cos^2 t}{1+\sin t} dt \\
&= \int \frac{1-\sin^2 t}{1+\sin t} dt \\
&= \int 1 - \sin t dt \\
&= t + \cos t + C.
\end{aligned}$$

Since $t = \arcsin x$, this is equal to $\arcsin x + \sqrt{1-x^2} + C$.