2021 - ISU Putnam Practice Set 6 - Solutions

Friday, October 28, 2022

Algebra

1. Let F be a finite field having an odd number m of elements. Let $p(x)$ be an irreducible (i.e. nonfactorable) polynomial over F of the form x^2+bx+c with $b,c \in F$. For how many elements $k \in F$ is $p(x)+k$ irreducible over F?

   Solution Putnam 1979 B3: The polynomial x^2+bx+d is reducible if and only if there are $s,t \in F$ such that

   $$st = d$$
   $$s+t = -b,$$

   or, equivalently, if and only if there is an element $s \in F$ with $-s(s+b) = d$. For $s \in F$, define $f(s) = s(s+b)$. We have just seen that the number of reducible polynomials of the form x^2+bx+d is equal to the number of elements in the image of f. Note that $f(s) = f(t)$ if and only if either $t = s$ or $t = -s-b$. Because the characteristic of F is not 2, $s = -s-b$ for only one $s \in F$. It follows that the image of F has $1+(m+1)/2$ distinct elements. Therefore the number of irreducible polynomials of the form $x^2+bx+c+k$ is $m-(1-(m-1)/2) = (m-1)/2$.

2. Let b and c be fixed real numbers and let the ten points (j,y_j) $j = 1,2,\dots,10$ lie on the parabola $y = x^2+bx+c$. For $j = 1,2,\dots,9$, let I_j be the point of intersection of the tangents to the given parabola at (j,y_j) and $(j+1,y_{j+1})$. Determine the polynomial function $y = g(x)$ of least degree who graph passes through all nine points I_j.

   Solution Putnam 1980 A1: We show that $g(x) = x^2+bx+c-(1/4)$. The equation of the tangent to the given parabola at $P_j = (j,y_j)$ is easily seen to be $y = L_j$, where $L_j = (2j+b)x-j^2+c$. Solving $y = L_j$ and $y = L_{j+1}$ simultaneously, one finds that $x = (2j+1)/2$ and so $j = (2x-1)/2$ at I_j. Substituting this expression for j into L_j gives the $g(x)$ above.

3. Find all polynomials of two variables satisfying

   $$P(a,b)P(c,d) = P(ac+bd, ad+bc)$$

for all real numbers a, b, c, d.

Solution: With a change of variables, we consider the polynomial

$$Q(x,y) = P\left(\frac{x+y}{2}, \frac{x-y}{2}\right).$$

Then

$$Q(x,y)Q(z,t) = Q(xz, yt).$$

Hence

$$Q(x,y) = Q(x,1)Q(1,y).$$

Since $Q(a,1)Q(b,1) = Q(ab,1)$ and $Q(1,c)Q(1,d) = Q(1,cd)$, the one variable polynomials $Q(x,1)$ and $Q(1,y)$ are multiplicative and must either be 0 or of the form x^n, y^m respectively. It follows that $P(x,y) = Q(x+y, x-y) = (x+y)^n(x-y)^m$ for some integers $m, n \geq 0$ or $P(x,y) = 0$.

4. Let n be a positive integer, and define

$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynoials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

for all $n \geq 1$. **Solution Putnam 1984 B1:** We have

$$f(n+2) - f(n+1) = (n+2)! = (n+2)(n+1)! = (n+2)[f(n+1) - f(n)].$$

It follows that we can take $P(x) = x+3$ and $Q(x) = -x-2$.

5. Prove or disprove the following statement: If F is a finite set with two or more elements, then there exists a binary operation $*$ on F such that for all $x, y, z \in F$:

   (a) $x * z = y * z$ implies $x = y$ (right cancelation holds), and

   (b) $x * (y * z) \neq (x * y) * z$ (no case of associativity holds).

Solution Putnam 1984 B3: The claim is true. Take any bijection ϕ of F with no fixed points and set $x * y = \phi(x)$.