2021 - ISU Putnam Practice Set 6 - Solutions

Wednesday, October 13, 2021

Polynomials

1. Find all solutions to the equation

$$(x+1)(x+2)(x+3)^2(x+4)(x+5) = 360.$$

Solution: Rewrite this as

$$(x^2+6x+5)(x^2+6x+8)(x^2+6x+9) = 360.$$

Substituting $y = x^2+6x$ we obtain

$$(y+5)(y+8)(y+9) = 360$$

or

$$y^3+22y^2+157y = 0,$$

which has solutions $y = 0, -11 \pm 6i$. If $y = 0$, then $x = 0$ or $x = -6$. If $y = -11+6i$, then we consider $x^2+6x = -11+6i$. This is equivalent to $(x+3)^2 = -2+6i$. Setting $x+3 = u+iv$ with $u, v \in \mathbb{R}$, we get a system of equations:

$$u^2 - v^2 = -2$$
$$2uv = 6.$$

It follows that $(u^2+v^2)^2 = (u^2-v^2)^2 + (2uv)^2 = 40$. Hence $u^2+v^2 = 2\sqrt{10}$. Then $u^2 = \sqrt{10}-1$, $v^2 = \sqrt{10}+1$. So $u = \pm\sqrt{\sqrt{10}-1}$, $v = \pm\sqrt{\sqrt{10}+1}$, and $x = u+iv-3$ for all four choices of signs for u and v.

2. Find all polynomials $P(x)$ satisfying the equation

$$(x+1)P(x) = (x-10)P(x+1).$$

Solution: The equation implies that $(x-10)$ divides $P(x)$. Replacing x by $x-1$ we obtain

$$xP(x-1) = (x-11)P(x)$$

showing that x also divides $P(x)$. So $P(x) = x(x-10)P_1(x)$ for some polynomial $P_1(x)$. Substituting into the original equation and canceling we get

$$xP_1(x) = (x-9)P_1(x+1).$$

Arguing as before, we find that $P_1(x) = (x-1)(x-9)P_2(x)$. Repeating the argument we eventually find that $P(x) = x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9)(x-10)Q(x)$, where $Q(x) = Q(x+1)$. It follows that $Q(x)$ is constant, and the solution to the problem is

$$P(x) = \alpha x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9)(x-10),$$

for any constant α.

3. Let $P(x)$ be a polynomial with complex coefficients. Prove that $P(x)$ is an even function if and only if there exists a polynomial $Q(x)$ with complex coefficients satisfying

$$P(x) = Q(x)Q(-x).$$

Solution: If such a Q exists, clearly $P(-x) = Q(-x)Q(x) = P(x)$ and so $P(x)$ is even.

Now suppose $P(x)$ is even. Identifying coefficients show that no odd powers appear in $P(x)$. Hence

$$P(x) = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdots + a_0 = P_1(x^2).$$

Factoring $P_1(y) = a(y-y_1)(y-y_2)\cdots(y-y_n)$, we have

$$P(x) = a(x^2-y_1)(x^2-y_2)\cdots(x^2-y_n).$$

Choosing complex numbers $b, x_1, \ldots, x_n$ such that $b^2 = (-1)^n a$ and $x_j^2 = y_j$, for $j = 1, \ldots, n$, we get the factorization:

$$\begin{aligned} P(x) &= b^2(x_1^2 - x^2)\cdots(x_n^2 - x^2) \\ &= (b(x_1 - x)\cdots(x_n - x))\,(b(x_1 + x)\cdots(x_n + x)) \\ &= Q(x)Q(-x), \end{aligned}$$

where $Q(x) = b(x_1 - x)\cdots(x_n - x)$.

4. Let $P(x)$ be a polynomial of degree n. Given that $P(k) = \frac{k}{k+1}$ for $k = 0, 1, \ldots, n$, find $P(m)$ for $m > n$.

   Solution: Because $P(0) = 0$, $P(x) = xQ(x)$ for some polynomial $Q(x)$. Then

   $$Q(k) = \frac{1}{k+1}, k = 1, 2, \ldots, n.$$

   Let $H(x) = (x+1)Q(x) - 1$. The degree of $H(x)$ is n and $H(k) = 0$ for $k = 1, 2, \ldots, n$. Hence

   $$H(x) = (x+1)Q(x) - 1 = a_0(x-1)(x-2)\cdots(x-n).$$

   Moreover, $H(-1) = -1$, yields $a_0 = \frac{(-1)^{n+1}}{(n+1)!}$. For $x = m$, $m > n$, which gives

   $$Q(m) = \frac{(-1)^{n+1}(m-1)(m-2)\cdots(m-n)+1}{(n+1)!(m+1)} + \frac{1}{m+1},$$

   and so

   $$P(m) = \frac{(-1)^{n+1}(m-1)(m-2)\cdots(m-n)}{(n+1)!(m+1)} + \frac{m}{m+1},$$

5. Let $Q_0(x) = 1$, $Q_1(x) = x$, and

   $$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

   for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

   Solution: 2017 (A2) Define $P_n(x)$ for $P_0(x) = 1$, $P_1(x) = x$, and $P_n(x) = xP_{n-1}(x) - P_{n-2}(x)$. We claim that $P_n(x) = Q_n(x)$ for all $n \geq 0$; since $P_n(x)$ clearly is a polynomial with integer coefficients for all n, this will imply the desired result.

   Since $\{P_n\}$ and $\{Q_n\}$ are uniquely determined by their respective recurrence relations and the initial conditions P_0, P_1 or Q_0, Q_1, it suffices to check that $\{P_n\}$ satisfies the same recurrence as Q: that is, $(P_{n-1}(x))^2 - P_n(x)P_{n-2}(x) = 1$ for all $n \geq 2$. Here is one proof of this: for $n \geq 1$, define the 2×2 matrices

   $$M_n = \begin{pmatrix} P_{n-1}(x) & P_n(x) \\ P_{n-2}(x) & P_{n-1}(x) \end{pmatrix}, \quad T = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}$$

   with $P_{-1}(x) = 0$ (this value being consistent with the recurrence). Then $\det(T) = 1$ and $TM_n = M_{n+1}$, so by induction on n we have

   $$(P_{n-1}(x))^2 - P_n(x)P_{n-2}(x) = \det(M_n) = \det(M_1) = 1.$$

Remark: A similar argument shows that any second-order linear recurrent sequence also satisfies a quadratic second-order recurrence relation. A familiar example is the identity $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ for F_n the n-th Fibonacci number. More examples come from various classes of *orthogonal polynomials*, including the Chebyshev polynomials mentioned below.

Second solution. We establish directly that $Q_n(x) = xQ_{n-1}(x) - Q_{n-2}(x)$, which again suffices. From the equation

$$1 = Q_{n-1}(x)^2 - Q_n(x)Q_{n-2}(x) = Q_n(x)^2 - Q_{n+1}(x)Q_{n-1}(x)$$

we deduce that

$$Q_{n-1}(x)(Q_{n-1}(x) + Q_{n+1}(x)) = Q_n(x)(Q_n(x) + Q_{n-2}(x)).$$

Since $\deg(Q_n(x)) = n$ by an obvious induction, the polynomials $Q_n(x)$ are all nonzero. We may thus rewrite the previous equation as

$$\frac{Q_{n+1}(x) + Q_{n-1}(x)}{Q_n(x)} = \frac{Q_n(x) + Q_{n-2}(x)}{Q_{n-1}(x)},$$

meaning that the rational functions $\frac{Q_n(x)+Q_{n-2}(x)}{Q_{n-1}(x)}$ are all equal to a constant value. By taking $n = 2$ and computing from the definition that $Q_2(x) = x^2 - 1$, we find the constant value to be x; this yields the desired recurrence.

Remark: By induction, one may also obtain the explicit formula

$$Q_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k}.$$

Remark: In light of the explicit formula for $Q_n(x)$, Karl Mahlburg suggests the following bijective interpretation of the identity $Q_{n-1}(x)^2 - Q_n(x)Q_{n-2}(x) = 1$. Consider the set C_n of integer compositions of n with all parts 1 or 2; these are ordered tuples $(c_1, \dots, c_k)$ such that $c_1 + \cdots + c_k = n$ and $c_i \in \{1,2\}$ for all i. For a given composition c, let $o(c)$ and $d(c)$ denote the number of 1's and 2's, respectively. Define the generating function

$$R_n(x) = \sum_{c \in C_n} x^{o(c)};$$

then $R_n(x) = \sum_j \binom{n-j}{j} x^{n-2j}$, so that $Q_n(x) = i^{-n/2} R_n(ix)$. (The polynomials $R_n(x)$ are sometimes called *Fibonacci polynomials*; they satisfy $R_n(1) = F_n$. This interpretation of F_n as the cardinality of C_n first arose in the study of Sanskrit prosody, specifically the analysis of a line of verse as a sequence of long and short syllables, at least 500 years prior to the work of Fibonacci.)

The original identity is equivalent to the identity

$$R_{n+1}(x)R_{n-1}(x) - R_n(x)^2 = (-1)^{n-1}.$$

This follows because if we identify the composition c with a tiling of a $1 \times n$ rectangle by 1×1 squares and 1×2 dominoes, it is *almost* a bijection to place two tilings of length n on top of each other, offset by one square, and hinge at the first possible point (which is the first square in either). This only fails when both tilings are all dominoes, which gives the term $(-1)^{n-1}$.

Remark: This problem appeared on the 2012 India National Math Olympiad; see `https://artofproblemsolving.com/community/c6h1219629`. Another problem based on the same idea is problem A2 from the 1993 Putnam.