Recurrence relations

1. Consider the sequence \((u_n)\) defined by \(u_0 = u_1 = u_2 = 1\), and

\[
\det \begin{pmatrix} u_{n+3} & u_{n+2} \\ u_{n+1} & u_n \end{pmatrix} = n! \quad \text{for } n \geq 0.
\]

Prove that \(u_n\) is an integer for all \(n\).

**Solution:** We prove by induction that \(u_n = (u - 1)(u - 3)(u - 5)\cdots\), interpreting an empty product as 1. This works for \(n = 0, 1, 2\) so assume that \(n \geq 3\) and the formula for \(u_n\) works for smaller \(n\). Then

\[
u_n = \frac{u_{n-1}u_{n-2} + (n - 3)!}{u_{n-3}}
= \frac{(n-2)(n-4)\cdots(n-3)(n-5)\cdots + (n-3)!}{(n-4)(n-6)\cdots}
= \frac{(n-2)! + (n-3)!}{(n-4)(n-6)\cdots}
= \frac{(n-2)(n-3)! + (n-3)!}{(n-4)(n-6)\cdots}
= \frac{(n-1)(n-3)!}{(n-4)(n-6)\cdots}
= \frac{(n-1)(n-3)(n-4)(n-5)(n-6)\cdots}{(n-4)(n-6)\cdots}
= (n-1)(n-3)(n-5)\cdots,
\]

as desired.

2. Find the general term of the sequence given by \(x_0 = 3, x_1 = 4\), and

\[(n + 1)(n + 2)x_n = 4(n + 1)(n + 3)x_{n-1} - 4(n + 2)(n + 3)x_{n-2}, \quad n \geq 2.\]

**Solution:** Divide through by the product \((n + 1)(n + 2)(n + 3)\). The recurrence relation becomes

\[
\frac{x_n}{n + 3} = 4 \frac{x_{n-1}}{n + 2} + 4 \frac{x_{n-2}}{n + 1}.
\]
The sequence \( y_n = \frac{x_n}{n+3} \) satisfies the recurrence
\[
y_n = 4y_{n-1} + 4y_{n-2}.
\]
 Its characteristic equation has the double root 2. Knowing that \( y_0 = 1 \) and \( y_1 = 1 \), we obtain \( y_n = 2^n - n2^{n-1} \). It follows that the answer to the problem is
\[
x_n = (n+3)2^n - n(n+3)2^{n-1}.
\]

3. Let \( x_0, x_1, x_2, \ldots \) be the sequence such that \( x_0 = 1 \) and for \( n \geq 0 \),
\[
x_{n+1} = \ln(e^{x_n} - x_n)
\]
(as usual, the function \( \ln \) is the natural logarithm). Show that the infinite series
\[
x_0 + x_1 + x_2 + \cdots
\]
converges and find its sum.

**Solution (Putnam 2016 B1):** Note that the function \( e^x - x \) is strictly increasing for \( x > 0 \) (because its derivative is \( e^x - 1 \), which is positive because \( e^x \) is strictly increasing), and its value at 0 is 1. By induction on \( n \), we see that \( x_n > 0 \) for all \( n \).

By exponentiating the equation defining \( x_{n+1} \), we obtain the expression
\[
x_n = e^{x_n} - e^{x_{n+1}}.
\]
We use this equation repeatedly to acquire increasingly precise information about the sequence \( \{x_n\} \).

- Since \( x_n > 0 \), we have \( e^{x_n} > e^{x_{n+1}} \), so \( x_n > x_{n+1} \).
- Since the sequence \( \{x_n\} \) is decreasing and bounded below by 0, it converges to some limit \( L \).
- Taking limits in the equation yields \( L = e^L - e^L \), whence \( L = 0 \).
- Since \( L = 0 \), the sequence \( \{e^{x_n}\} \) converges to 1.

We now have a telescoping sum:
\[
x_0 + \cdots + x_n = (e^{x_0} - e^{x_1}) + \cdots + (e^{x_n} - e^{x_{n+1}})
\]
\[
= e^{x_0} - e^{x_{n+1}} = e - e^{x_{n+1}}.
\]
By taking limits, we see that the sum \( x_0 + x_1 + \cdots \) converges to the value \( e - 1 \).
4. The sequence \( (x_n) \) is defined by \( x_1 = 4, x_2 = 19, \) and for \( n \geq 2, \)

\[
x_{n+1} = \left\lceil \frac{x_n^2}{x_{n-1}} \right\rceil.
\]

the smallest integer greater than or equal to \( \frac{x_n^2}{x_{n-1}} \). Prove that \( x_n - 1 \) is always a multiple of 3.

**Solution:** We compute \( x_3 = 91, x_4 = 436, x_5 = 2089. \) Let us hope that the terms of the sequence satisfy a recurrence \( x_{n+1} = \alpha x_n + \beta x_{n-1}. \) Substituting \( n = 2 \) and \( n = 3 \) we obtain \( \alpha = 5, \beta = -1, \) and then the relation is also verified for the next term \( 2089 = 5 \cdot 436 - 91. \)

Let us prove that this recurrence holds in general.

If \( y_n \) is the general term of this recurrence, then \( y_n = ar^n + bs^n, \) where \( r = 5 + \sqrt{21}, s = 5 - \sqrt{21}, rs = 1, r - s = \sqrt{21}; \) and \( a = \frac{7 + \sqrt{21}}{14}, b = \frac{7 - \sqrt{21}}{14}, ab = 1. \) We then compute

\[
y_{n+1} - \frac{y_n^2}{y_{n-1}} = \frac{y_{n+1}y_{n-1} - y_n^2}{y_{n-1}} = \frac{(ar^{n+1} + bs^{n+1})(ar^{n-1} + bs^{n-1}) - (ar^n + bs^n)^2}{ar^{n-1} + bs^{n-1}} = \frac{ab(rs)^{n-1}(r-s)^2}{y_{n-1}} = \frac{3}{y_{n-1}}.
\]

Clearly \( 0 < \frac{3}{y_{n-1}} < 1 \) for \( n \geq 2. \) Because \( y_{n+1} \) is an integer, it follows that

\[
y_{n+1} = \left\lfloor \frac{y_n^2}{y_{n-1}} \right\rfloor.
\]

Hence \( x_n \) and \( y_n \) satisfy the same recurrence. This implies that \( x_n = y_n \) for all \( n. \) The conclusion follows by induction if we write

\[
(x_{n+1} - 1) = 5(x_n - 1) - (x_{n-1} - 1) + 3.
\]

5. Prove that every nonzero coefficient of the Taylor series of

\[
(1 - x + x^2)e^x
\]

about \( x = 0 \) is a rational number whose numerator (in lowest terms) is either 1 or a prime number.
Solution (Putnam 2014 A1): The coefficient of $x^n$ in the Taylor series of $(1 - x + x^2)e^x$ for $n = 0, 1, 2$ is $1, 0, \frac{1}{2}$, respectively. For $n \geq 3$, the coefficient of $x^n$ is

$$\frac{1}{n!} - \frac{1}{(n-1)!} + \frac{1}{(n-2)!} = \frac{1-n+n(n-1)}{n!} = \frac{n-1}{n(n-2)!}.$$  

If $n - 1$ is prime, then the lowest-terms numerator is clearly either 1 or the prime $n - 1$ (and in fact the latter, since $n - 1$ is relatively prime to $n$ and to $(n-2)!$). If $n - 1$ is composite, either it can be written as $ab$ for some $a \neq b$, in which case both $a$ and $b$ appear separately in $(n-2)!$ and so the numerator is 1, or $n - 1 = p^2$ for some prime $p$, in which case $p$ appears in $(n-2)!$ and so the numerator is either 1 or $p$. (In the latter case, the numerator is actually 1 unless $p = 2$, as in all other cases both $p$ and $2p$ appear in $(n-2)!$.)