2023 - ISU Putnam Practice Set 5 - Solutions

Thursday, October 5, 2023

Recurrence relations

1. Consider the sequence (u_n) defined by $u_0 = u_1 = u_2 = 1$, and

$$\det \begin{pmatrix} u_{n+3} & u_{n+2} \\ u_{n+1} & u_n \end{pmatrix} = n! \qquad \text{for } n \geq 0.$$

Prove that u_n is an integer for all n.

Solution: We prove by induction that $u_n = (u-1)(u-3)(u-5)\cdots$, interpretting an empty product as 1. This works for $n = 0, 1, 2$ so assume that $n \geq 3$ and the formula for u_n works for smaller n. Then

$$\begin{aligned}
u_n &= \frac{u_{n-1}u_{n-2} + (n-3)!}{u_{n-3}} \\
&= \frac{(n-2)(n-4)\cdots(n-3)(n-5)\cdots + (n-3)!}{(n-4)(n-6)\cdots} \\
&= \frac{(n-2)! + (n-3)!}{(n-4)(n-6)\cdots} \\
&= \frac{(n-2)(n-3)! + (n-3)!}{(n-4)(n-6)\cdots} \\
&= \frac{(n-1)(n-3)!}{(n-4)(n-6)\cdots} \\
&= \frac{(n-1)(n-3)(n-4)(n-5)(n-6)\cdots}{(n-4)(n-6)\cdots} \\
&= (n-1)(n-3)(n-5)\cdots,
\end{aligned}$$

as desired.

2. Find the general term of the sequence given by $x_0 = 3$, $x_1 = 4$, and

$$(n+1)(n+2)x_n = 4(n+1)(n+3)x_{n-1} - 4(n+2)(n+3)x_{n-2}, \qquad n \geq 2.$$

Solution: Divide through by the product $(n+1)(n+2)(n+3)$. The recurrence relation becomes

$$\frac{x_n}{n+3} = 4\frac{x_{n-1}}{n+2} + 4\frac{x_{n-2}}{n+1}.$$

The sequence $y_n = \frac{x_n}{n+3}$ satisfies the recurrence

$$y_n = 4y_{n-1} + 4y_{n-2}.$$

Its characteristic equation has the double root 2. Knowing that $y_0 = 1$ and $y_1 = 1$, we obtain $y_n = 2^n - n2^{n-1}$. It follows that the answer to the problem is

$$x_n = (n+3)2^n - n(n+3)2^{n-1}.$$

3. Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.

Solution (Putnam 2016 B1): Note that the function $e^x - x$ is strictly increasing for $x > 0$ (because its derivative is $e^x - 1$, which is positive because e^x is strictly increasing), and its value at 0 is 1. By induction on n, we see that $x_n > 0$ for all n.

By exponentiating the equation defining x_{n+1}, we obtain the expression

$$x_n = e^{x_n} - e^{x_{n+1}}.$$

We use this equation repeatedly to acquire increasingly precise information about the sequence $\{x_n\}$.

- Since $x_n > 0$, we have $e^{x_n} > e^{x_{n+1}}$, so $x_n > x_{n+1}$.
- Since the sequence $\{x_n\}$ is decreasing and bounded below by 0, it converges to some limit L.
- Taking limits in the equation yields $L = e^L - e^L$, whence $L = 0$.
- Since $L = 0$, the sequence $\{e^{x_n}\}$ converges to 1.

We now have a telescoping sum:

$$\begin{aligned} x_0 + \cdots + x_n &= (e^{x_0} - e^{x_1}) + \cdots + (e^{x_n} - e^{x_{n+1}}) \\ &= e^{x_0} - e^{x_{n+1}} = e - e^{x_{n+1}}. \end{aligned}$$

By taking limits, we see that the sum $x_0 + x_1 + \cdots$ converges to the value $e - 1$.

4. The sequence (x_n) is defined by $x_1 = 4$, $x_2 = 19$, and for $n \geq 2$,

$$x_{n+1} = \left\lceil \frac{x_n^2}{x_{n-1}} \right\rceil.$$

the smallest integer greater than or equal $\frac{x_n^2}{x_{n-1}}$. Prove that $x_n - 1$ is always a multiple of 3.

Solution: We compute $x_3 = 91$, $x_4 = 436$, $x_5 = 2089$. Let us hope that the terms of the sequence satisfy a recurrence $x_{n+1} = \alpha x_n + \beta x_{n-1}$. Substituting $n = 2$ and $n = 3$ we obtain $\alpha = 5$, $\beta = -1$, and then the relation is also verified for the next term $2089 = 5 \cdot 436 - 91$. Let us prove that this recurrence holds in general.

If y_n is the general term of this recurrence, then $y_n = ar^n + bs^n$, where $r = 5 + \sqrt{21}$, $s = 5 - \sqrt{21}$, $rs = 1$, $r - s = \sqrt{21}$; and $a = \frac{7+\sqrt{21}}{14}$, $b = \frac{7-\sqrt{21}}{14}$, $ab = 1$. We then compute

$$\begin{aligned} y_{n+1} - \frac{y_n^2}{y_n - 1} &= \frac{y_{n+1}y_{n-1} - y_n^2}{y_{n-1}} \\ &= \frac{(ar^{n+1} + bs^{n+1})(ar^{n-1} + bs^{n-1}) - (ar^n + bs^n)^2}{ar^{n-1} + bs^{n-1}} \\ &= \frac{ab(rs)^{n-1}(r-s)^2}{y_{n-1}} \\ &= \frac{3}{y_n - 1}. \end{aligned}$$

Clearly $0 < \frac{3}{y_n - 1} < 1$ for $n \geq 2$. Because y_{n+1} is an integer, it follows that

$$y_{n+1} = \left\lceil \frac{y_n^2}{y_{n-1}} \right\rceil.$$

Hence x_n and y_n satisfy the same recurrence. This implies that $x_n = y_n$ for all n. The conclusion follows by induction if we write

$$(x_{n+1} - 1) = 5(x_n - 1) - (x_{n-1} - 1) + 3.$$

5. Prove that every nonzero coefficient of the Taylor series of

$$(1 - x + x^2)e^x$$

about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

Solution (Putnam 2014 A1): The coefficient of x^n in the Taylor series of $(1-x+x^2)e^x$ for $n = 0,1,2$ is $1,0,\frac{1}{2}$, respectively. For $n \geq 3$, the coefficient of x^n is

$$\begin{aligned} \frac{1}{n!} - \frac{1}{(n-1)!} + \frac{1}{(n-2)!} &= \frac{1-n+n(n-1)}{n!} \\ &= \frac{n-1}{n(n-2)!}. \end{aligned}$$

If $n-1$ is prime, then the lowest-terms numerator is clearly either 1 or the prime $n-1$ (and in fact the latter, since $n-1$ is relatively prime to n and to $(n-2)!$). If $n-1$ is composite, either it can be written as ab for some $a \neq b$, in which case both a and b appear separately in $(n-2)!$ and so the numerator is 1, or $n-1 = p^2$ for some prime p, in which case p appears in $(n-2)!$ and so the numerator is either 1 or p. (In the latter case, the numerator is actually 1 unless $p = 2$, as in all other cases both p and $2p$ appear in $(n-2)!$.)