# 2021 - ISU Putnam Practice Set 5 - Solutions

## Wednesday, October 21, 2022

## Geometry

1. Prove that the midpoints of the sides of a quadrilateral form a parallelogram.

   **Solution:** After an affine transformation, we can assume one vertex, labeled A, is at the origin. Denote the vectors $u = \overrightarrow{AB}$, $v = \overrightarrow{BC}$, $w = \overrightarrow{AD}$, and $x = \overrightarrow{DC}$. Then the midpoints are the points with coordinates $a = \frac{1}{2}u$, $b = u + \frac{1}{2}v$, $c = \frac{1}{2}w$ and $d = w + \frac{1}{2}x$. To show that $\overrightarrow{ab}$ is parallel to $\overrightarrow{cd}$ we compute:

   $$\overrightarrow{ab} = b - a = (u + \frac{1}{2}v) - \frac{1}{2}u = \frac{1}{2}(u + v)$$

   and

   $$\overrightarrow{cd} = d - c = (w + \frac{1}{2}x) - \frac{1}{2}w = \frac{1}{2}(w + x).$$

   Since $ABCD$ is a quadrilateral, $u + v = w + x$, and hence $\overrightarrow{ab} = \overrightarrow{cd}$. A similar calculation shows the other two edges are parallel.

2. Given any 9 lattice points in space, show that we can find two which have a lattice point on the interior of the segment joining them.

   **Solution Putnam 1971 A1:** We can divide the points into 8 categories according to the parity of each coordinate. There must be at least 2 points in the same category. The midpoint of the line joining them is then also a lattice point.

3. Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, $A$, $B$, and $C$, which are vertices of an equilateral triangle, and find its area.

   **Solution Putnam 2006 B1:** The "curve" $x^3 + 3xy + y^3 - 1 = 0$ is actually reducible, because the left side factors as

   $$(x + y - 1)(x^2 - xy + y^2 + x + y + 1).$$

   Moreover, the second factor is

   $$\frac{1}{2}((x+1)^2 + (y+1)^2 + (x-y)^2),$$

so it only vanishes at $(-1,-1)$. Thus the curve in question consists of the single point $(-1,-1)$ together with the line $x+y=1$. To form a triangle with three points on this curve, one of its vertices must be $(-1,-1)$. The other two vertices lie on the line $x+y=1$, so the length of the altitude from $(-1,-1)$ is the distance from $(-1,-1)$ to $(1/2,1/2)$, or $3\sqrt{2}/2$. The area of an equilateral triangle of height $h$ is $h^2\sqrt{3}/3$, so the desired area is $3\sqrt{3}/2$.

**Remark:** The factorization used above is a special case of the fact that

$$\begin{aligned} &x^3+y^3+z^3-3xyz \\ &= (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z), \end{aligned}$$

where $\omega$ denotes a primitive cube root of unity. That fact in turn follows from the evaluation of the determinant of the *circulant matrix*

$$\begin{pmatrix} x & y & z \\ z & x & y \\ y & z & x \end{pmatrix}$$

by reading off the eigenvalues of the eigenvectors $(1,\omega^i,\omega^{2i})$ for $i=0,1,2$.

4. A convex polygon does not extend outside a square side 1. Prove that the sum of the squares of its sides is at most 4.

   **Solution Putnam 1966 B1:** Form a right-angled triangle on each side of the polygon (and outside it), by taking the other two sides parallel to the sides of the square. The sum of the squares of the polygon's sides equals the sum of the squares of the non-hypoteneuse sides of the triangles. Because the polygon is convex, these triangle sides form 4 sets, one for each side of the square, and each set having lengths totaling less than 1 (the side of the square). So the sum of the squares in each set is less than $\sum x_i^2 < (\sum x_i)^2 = 1$.

5. The vertices of a triangle are lattice points in the plane. Show that the diameter of its circumcircle does not exceed the product of its side lengths.

   **Solution Putnam 1971 A3:** Let the side lengths be a, b, c and the circumradius R. Let $\theta$ be the angle opposite side a. Then the area of the triangle $A = 1/2bc\sin\theta$. The side a subtends an angle $2\theta$ at the centre of the circumcircle, so $a = 2R\sin\theta$. Hence $2A = abc/(2R)$. So we have to show that $A \geq 1/2$.

This follows at once from the well-known Pick's theorem: the area of any (non-self-intersecting) polygon whose vertices are lattice points is $v/2+i-1$, where $v$ is the number of lattice points on the perimeter and i is the number of lattice points inside the polygon (so since for a triangle $v \geq 3$, and $i \geq 0$, we have area at least $3/2-1=1/2$).