2021 - ISU Putnam Practice Set 5 - Solutions

Wednesday, October 6, 2021

Even More Calculus

1. Compute

$$\int_0^{\sqrt{\pi/3}} \sin(x^2)dx + \int_{-\sqrt{\pi/3}}^{\sqrt{\pi/3}} x^2\cos(x^2)dx.$$

Solution: Since $x^2\cos(x^2)$ is an even function, we get

$$\begin{aligned}\int_0^{\sqrt{\pi/3}} \sin(x^2)dx + \int_{-\sqrt{\pi/3}}^{\sqrt{\pi/3}} x^2\cos(x^2)dx &= \int_0^{\sqrt{\pi/3}} \sin(x^2)+2x^2\cos(x^2)dx \\ &= \int_0^{\sqrt{\pi/3}} \frac{d}{dx}\left(x\sin(x^2)\right)dx \\ &= x\sin(x^2)\big|_0^{\sqrt{\pi/3}} = \frac{\pi}{2}.\end{aligned}$$

2. Compute

$$I_1 = \int \frac{\sin x}{\sin x+\cos x}dx.$$

Solution: Set $I_2 = \int \frac{\cos x}{\sin x+\cos x}dx$ so that

$$\begin{aligned}I_1+I_2 &= \int \frac{\sin x+\cos x}{\sin x+\cos x}dx = \int 1dx = x+C_1 \\ I_2-I_1 &= \int \frac{\cos x-\sin x}{\sin x+\cos x}dx = \ln|\sin x+\cos x|+C_2\end{aligned}$$

Therefore $I_1 = \frac{1}{2}[(I_1+I_2)-(I_2-I_1)] = \frac{x}{2} - \frac{\ln|\sin x+\cos x|}{2} + C'$.

3. Evaluate

$$I = \int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}.$$

Solution (1987 B1): Use the substitution $x = 6-y$ to see that

$$I = \int_2^4 \frac{\sqrt{\ln(y+3)}}{\sqrt{\ln(y+3)}+\sqrt{\ln(9-y)}}dy.$$

Therefore

$$2I = \int_2^4 \frac{\sqrt{\ln(9-y)}+\sqrt{\ln(y+3)}}{\sqrt{\ln(y+3)}+\sqrt{\ln(9-y)}}dy = \int_2^4 dy = 2,$$

and $I = 1$.

4. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx = 1.$$

Show that

$$\int_0^1 (f(x))^2 dx \geq 4.$$

Solution: Note that if $g(x) = 6x-2$, then

$$\int_0^1 g(x)dx = \int_0^1 xg(x)dx = 1.$$

So

$$\int_0^1 (f(x)-g(x))dx = \int_0^1 x(f(x)-g(x))dx = 0,$$

and so

$$\int_0^1 g(x)(f(x)-g(x))dx = 6\int_0^1 x(f(x)-g(x))dx - 2\int_0^1 (f(x)-g(x))dx = 0.$$

Then

$$\begin{aligned} 0 \leq \int_0^1 (f(x)-g(x))^2 dx &= \int_0^1 f(x)(f(x)-g(x))dx - \int_0^1 g(x)(f(x)-g(x))dx \\ = \int_0^1 (f(x))^2 - f(x)g(x)dx - 0 &= \int_0^1 f^2(x)dx - 6\int_0^1 xf(x)dx + 2\int_0^1 f(x)dx \\ = \int_0^1 (f(x))^2 dx - 4. \end{aligned}$$

5. Compute the integral

$$I = \int_0^\pi \frac{x\sin x}{1+\sin^2 x}dx.$$

Solution: Consider the substitution $t = \pi - x$ so that $\sin(x) = \sin(t)$. Then the above integral becomes

$$\int_0^\pi \frac{x\sin x}{1+\sin^2 x}dx = \int_0^\pi \frac{(\pi - t)\sin t}{1+\sin^2 t}dt.$$

Changing the t to x and adding gives us

$$2I = \pi\int_0^\pi \frac{\sin x}{1+\sin^2 x}dx = \pi\int_0^\pi \frac{\sin x}{2-\cos^2 x}dx.$$

After the substitution $u = \cos(x)$, this becomes

$$2I = \pi \int_{-1}^{1} \frac{1}{2-u^2} du = \frac{\pi}{2\sqrt{2}} \int_{-1}^{1} \left(\frac{1}{\sqrt{2}-u} + \frac{1}{\sqrt{2}+u} \right) du = \frac{\pi}{2\sqrt{2}} \ln\left(\frac{\sqrt{2}+u}{\sqrt{2}-u} \right) \Bigg|_{0}^{1} = \frac{\pi}{2\sqrt{2}} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right).$$

Therefore

$$I = \frac{\pi}{2\sqrt{2}} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right)$$