2023 - ISU Putnam Practice Set 4 - Solutions

Thursday, September 28, 2023

Calculus 1

1. Let $f(x) = a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_n \sin(nx)$, where $a_1, a_2, \ldots, a_n$ are real numbers and n is a positive integer. Given that $|f(x)| \le |\sin(x)|$ for all real x, prove that

$$|a_1 + 2a_2 + \cdots na_n| \le 1.$$

Solution (Putnam 1967 A-1):

$$\begin{aligned}
|a_1 + 2a_2 + \cdots + na_n| &= |f'(0)| \\
&= \lim_{x\to 0}\left|\frac{f(x) - f(0)}{x}\right| \\
&= \lim_{x\to 0}\left|\frac{f(x)}{\sin(x)}\right| \cdot \left|\frac{\sin(x)}{x}\right| \\
&= \lim_{x\to 0}\left|\frac{f(x)}{\sin(x)}\right| \le 1.
\end{aligned}$$

2. Prove that not all zeros of the polynomial $P(x) = x^4 - \sqrt{7}x^3 + 4x^2 - \sqrt{22}x + 15$ are real.

Solution: If all four zeros of the polynomial $P(x)$ are real, then by Rolle's theorem all three zeros of $P'(x)$ are real, and consequently both zeros of $P''(x) = 12x^2 - 6\sqrt{7}x + 8$ are real. But this quadratic polynomial has the discriminant equal to -132, which is negative, and so it has complex zeros. The contradiction implies that not all zeros of $P(x)$ are real.

3. For any real number $\lambda \ge 1$, denote by $f(\lambda)$ the real solution to the equation

$$x(1 + \ln x) = \lambda.$$

Prove that

$$\lim_{\lambda\to\infty} \frac{f(\lambda)}{\lambda/\ln(\lambda)} = 1.$$

Solution: The function $h : [1,\infty) \to [1,\infty)$ given by $h(t) = t(1 + \ln t)$ is strictly increasing, and $h(1) = 1$, $\lim_{t\to\infty} h(t) = \infty$. Hence h is bijective, and its inverse is clearly the function

$f : [1,\infty) \to [1,\infty)$, and $\lambda \to f(\lambda)$. Since h is differentiable, so is f, and

$$f'(\lambda) = \frac{1}{f'(h(\lambda))} = \frac{1}{2+\ln(f(\lambda))}.$$

Also, since h is strictly increasing and $\lim_{t\to\infty} h(t) = \infty$, $f(\lambda)$ is strictly increasing, and its limit at infinity is also infinity. Using the defining relation for $f(\lambda)$, we see that

$$\frac{f(\lambda)}{\lambda/\ln(\lambda)} = \ln(\lambda)\cdot\frac{f(\lambda)}{\lambda} = \frac{\ln(\lambda)}{1+\ln(f(\lambda))}.$$

Now we apply L'Hopital's theorem and obtain

$$\begin{aligned}
\lim_{\lambda\to\infty}\frac{f(\lambda)}{\lambda/\ln(\lambda)} &= \lim_{\lambda\to\infty}\frac{\ln(\lambda)}{1+\ln(f(\lambda))}\\
&= \lim_{\lambda\to\infty}\frac{1/\lambda}{f'(\lambda)/f(\lambda)}\\
&= \lim_{\lambda\to\infty}\frac{f(\lambda)}{\lambda}(2+\ln(f(\lambda)))\\
&= \lim_{\lambda\to\infty}\frac{2+\ln(f(\lambda))}{1+\ln(f(\lambda))}\\
&= 1.
\end{aligned}$$

where the next-to-last equality follows again from $f(\lambda)(1+\ln f(\lambda)) = \lambda$. Therefore, the required limit is equal to 1.

4. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n)-f(x)}{n}$$

for all real numbers x and all positive integers n.

Solution (Putnam 2010 A-2): The only such functions are those of the form $f(x) = cx+d$ for some real numbers c,d (for which the property is obviously satisfied). To see this, suppose that f has the desired property. Then for any $x \in \mathbb{R}$,

$$\begin{aligned}
2f'(x) &= f(x+2)-f(x)\\
&= (f(x+2)-f(x+1)) + (f(x+1)-f(x))\\
&= f'(x+1)+f'(x).
\end{aligned}$$

Consequently, $f'(x+1) = f'(x)$.

Define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x+1) - f(x)$, and put $c = g(0)$, $d = f(0)$. For all $x \in \mathbb{R}$, $g'(x) = f'(x+1) - f'(x) = 0$, so $g(x) = c$ identically, and $f'(x) = f(x+1) - f(x) = g(x) = c$, so $f(x) = cx + d$ identically as desired.

5. For each continuous function $f : [0,1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)\,dx$ and $J(x) = \int_0^1 x\,(f(x))^2\,dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

   Solution (Putnam 2006 B-5: The answer is $1/16$. We have

$$\begin{aligned}
&\int_0^1 x^2 f(x)\,dx - \int_0^1 x f(x)^2\,dx \\
&= \int_0^1 (x^3/4 - x(f(x) - x/2)^2)\,dx \\
&\leq \int_0^1 x^3/4\,dx = 1/16,
\end{aligned}$$

   with equality when $f(x) = x/2$.